IDENTICAL DUAL LATTICES AND SUBDIVISION OF SPACE

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This book is based on a PhD thesis which was submitted to the Technion Israeli Institute of technology on June 2003, as well as additional research made since then.

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1 Abstract

The issue of partitioning space underlies the architectural planning and design of structures and spaces allocated for human activity.

This thesis focuses on the phenomenon of periodic dual spaces and the partition between them. Periodic three-dimensional networks can represent the inner structure of these spaces. Each periodic network in space has only one dual network (dual networks are discussed in the thesis). These network pairs are often referred to as complementary or reciprocal networks. Every dual network pair can be partitioned and separated by a smooth hyperbolic surface.

The thesis explores the unique phenomenon of identical dual networks and the hyperbolic surface separating them and dividing the space into two identical subspaces.

The purpose of the thesis is to investigate the phenomenon of identical dual spaces, the relations between the order and organization of the space, and the nature of the resultant symmetry operations, and the partitioning of the space into two identical subspaces and the nature of the dividing 2-manifold between them. An additional goal we set ourselves at the outset of the thesis was to examine the likelihood range of the phenomenon according to its definitions, and to find a way to classify and exhaust the surfaces dividing the space into two identical subspaces.

The thesis comprises four stages:

The first stage consisted of studying identical dual spaces, their properties and significance.

The tunnel-like periodic spaces represented by networks are defined within the Euclidean three-dimensional space, and are

composed of periodic cells. This property indicates the relation between the networks and the surface separating them and the symmetry groups acting on this space.

The smallest periodic cell of the periodic space derived from that space by the symmetry operation of the symmetry group acting on this space is entitled “The Elementary Periodic Region” (EPR). The EPR exemplifies all the properties of the space and all its phenomena, such as representation of the complementary dual networks, the partition surface separating them, and the symmetry operation.

At this stage of the study the topological properties of the smooth 2-manifolds in general and those of smooth 2-manifolds dividing the space between two dual networks in particular were explored.

The major properties of these 2-manifolds are:

a. The 2-manifolds are smooth, periodic and hyperbolic, and exist in the three-dimensional space.
b. The 2-manifolds divide the space into two identical sub-spaces that graphically represent two transposing tunneled networks which do not intersect. The tunnel networks are identical in volume and shape, and their axes form two identical three-dimensional networks.
c. The 2-fold 1800 rotation axes contained in the 2-manifolds rotate one sub-space into the identical complementary subspace, forming a periodic three-dimensional network referred to in this thesis as “a 2-fold network”.

An unequivocal distinction was made between minimal hyperbolic surfaces that can be physically realized by dipping a defined perimeter in soap water (henceforward “soap membrane networks”) and minimal surfaces whose dipping is not physically solvable.

The second stage of the thesis focused on the development of a method to enumerate and classify the 2-manifolds that divide the space into two identical subspaces, and to identify the axes of the identical dual tunnel networks. The method of enumerating the 2-manifolds was based on the topological properties of the 2-manifolds studied in the first stage.

It is noteworthy that the issue of periodic minimal hyperbolic 2-manifolds that divide the space into two identical subspaces were investigated in the past with special regard to the first category of minimal realizable surfaces (“soap membrane networks”). Seven surfaces corresponding to these characteristics were found. The second category, that of periodic minimal surfaces which cannot be physically realized by dipping the perimeter in soap water, was investigated in connection with the division of the space into two identical subspaces in a very preliminary manner. It was clear even then that numerous identical networks and a smooth hyperbolic partition dividing them could be put through every “elementary three-dimensional space cell” containing 2-fold axes. No attempt has been made to date either to construct and characterize these surfaces or to define the range of their existence and exhaust it.

The existence and periodicity of the 2-manifolds that divide the space into two identical subspaces in the Euclidean three-dimensional space indicate the link between these 2-manifolds and the symmetry groups operating within this space.

The “atomistic” conception of the periodic space suggests the existence of an EPR (elementary periodic region) that embodies all the properties of the periodic space. Locating the “elementary periodic regions” that represent identical dual spaces (elementary regions in which the axes of the 2-fold network is represented) and the dividing surface leads to the discovery of the 2-manifolds.

Because the number of the E.P.R.s representing the symmetry groups operating in the Euclidean space is finite, the number of E.P.R.s is also finite

The method of enumerating the 2-manifolds developed in the thesis consists of consecutive steps described below. Each step narrows down the scrutinized area and brings us closer to the final goal:

a. Locating all the E.P.R.s derived from the Euclidean space by the symmetry groups.
b. Locating in the E.P.R.s enumerated in the first stage the E.P.R.s representing the networks whose rotation axes contained in these networks, namely E.P.R.s containing 2-fold networks that can rotate the E.P.R.s into themselves.
c. Locating 2-fold networks by duplicating the E.P.R.s found in step 2.
d. Locating periodic cells enclosed in the 2-fold network (more than one is likely), which may enclose a periodic unit of the 2-manifold, and exhaustion of the possible 2-manifold units.
e. Duplicating the periodic cells with the enclosed periodic unit of the 2-manifolds, until a large enough section of the surface enabling identification of the self-dual tunnel network is obtained.
f. Topological enumeration of each 2-manifolds according to its respective tunnel network.

The third stage focused on the process of locating the 2-fold networks.

Locating the rotation axes of the networks is supposed to be exhausted at this stage. The method of locating the 2-fold networks is based on the identification of the E.P.R.s that may, as stated above, contain axes that rotate them into themselves. This is not a systematic or combinatorial method and hence we may have missed some 2-fold networks. At this stage, fourteen different 2-fold networks were found. Twelve of the fourteen 2-fold networks were found to contain seventeen different closed periodic cells that may enclose a periodic unit of a 2-manifold network entitled “Elementary Periodic Segment”.

With regard to the first category, duplicating these closed cells – “the soap membrane networks” – led to the discovery of eight topologically different 2-manifolds so far. Among them, a new, unknown to date, 2-manifold was discovered, which was designated “The Cubic Diamond 2-Manifold.”

Among the seventeen closed cells mentioned above, there are six closed cells with a split perimeter. These split perimeters can enclose “elementary periodic segments” of the second category, i.e. they can enclose in the periodic cell a successive smooth hyperbolic “elementary periodic segment” which is topologically different from the minimal “elementary periodic segments” previously discovered. Thus, a new class of 2-manifolds designated “The Multiple-Sleeved Class” was found. This class consists of an infinite number of topologically different 2-manifolds. This indicates that the tunnel axes of these 2-manifolds represent pairs of identical-dual networks that differ from each other.

The method of locating the 2-manifolds that divide pairs of identical-dual subspaces proposed in this dissertation led to the discovery of new 2-manifolds in addition to the seven 2-manifolds found by Prof. M. Burt. The 2-manifolds were looked for in representative groups, such as the elementary regions and the symmetry groups, which have a finite number of members. This does not necessarily mean that the number of 2-manifolds and/or the number of identical-dual pairs of networks is also finite.

Following are a few suggestions about how to further pursue this study:

a. Locating other 2-fold networks apart from the fourteen that were found. We have no proof of exhaustion, and it is likely that other networks exist. Looking for these networks in the course of implementing the method proposed in this study of locating and enumerating the 2-manifolds, was pursued by identifying the elementary regions containing an axis or axes that rotate the 2-manifolds into themselves. It is likely that some of the elementary 2-manifolds have eluded us.
b. Locating closed three-dimensional cells, apart from the seventeen that were found. Because we have no evidence of exhaustion, these cells may be located in the new 2-fold networks that may be discovered, as well as in the 2-fold networks found in out study.
c. Locating new 2-manifold classes in addition to the ones mentioned in out thesis:
   - The “soap membrane networks” class.
   - The “multiple-sleeved networks” class.

In addition to locating and classifying the 2-manifolds, this dissertation proposes a method for designating the dual spaces. The dual spaces, or the tunnel networks representing these spaces, characterize topologically the 2-manifolds separating them. Designating the networks means designating the 2-manifolds. The method of designating the networks is based on the fact that each network contains a number of typical “packaging cells”.

Designating the packaging cells leads to the identification of the networks and of the dual spaces and hence to the identification of the 2-manifolds dividing them. The notation of the packaging cells is based on the notation method developed for the Platonic and Archimedean solids, adapted to irregular packaging solids, to solids with hyperbolic faces, to solids with irregular polygon faces, and to solids with non-uniform vertices. This notation method does not purport to designate all the possible solids: the more complex the solid the more sophistication required to interpret it. Moreover, interpretation may sometimes be impossible. The proposed method is therefore suitable for the simplest cases. More complex cases may have to resort to literary notations.

2 Introduction

2.1 The research and objectives of the dissertation

> **In the beginning God created the heavens and the earth. Now the earth was formless and empty, darkness was over the surface of the deep… and he separated the light from the darkness… and separated the water under the vault from the water above it… God called the dry ground "land," and the gathered waters he called "seas."… (Genesis 1)**

The partition of space lies at the core of architectural design. The definition of regions, their organization, and their relations. Both the private and public spaces, from personal living spaces, through streets, yards, and public buildings, are defined by partitions. Some physical, and some virtual and ethereal.

These partitions, which separate different spaces, often describe open and continuous envelopes, otherwise we would have closed spaces, with no entrance or exit, which would hence be unusable. The continuity of partitions allows the connection between the spaces, and the transition from region to region in a form of labyrinth.

The labyrinth may be ordered in different degrees of order. When talking about degrees of order, the opposite of chaos, we're talking about degrees of repetition. Meaning, there are repeating elements of different levels of complexity. For instance, a room in a living structure, a room in an office building, a room in a hotel, etc. From the urban perspective, it is possible to observe a lot, structure, neighborhood, city, etc. as repeating elements.

Since the beginning there has not existed chaos. We can always find some order in any phenomenon we examine, from the distinction between light and darkness, to the nucleus of the atom.

The partitions which define multiple subspaces in the three-dimensional space create a form of duality between the subspaces. That is, there is a link between the subspace from one side of the partition to the dual subspace on the other side of the partition. In ordered and cyclical arrays, this duality manifests through exact patterns.

The morphological-conceptual discussion in cyclical two-dimensional partitions is of great importance for understand space and the ability to develop an ability to understand and organize space, both functionally and from a design perspective.

The phenomenon of partitioning space into subspaces is a widespread phenomenon. It is possible to observe multiple methods for partitioning space. The ones most commonly known are:

1. The partitioning of space by envelopes which define closed spaces. In this case, the morphological discussion deals with the relation between the spaces, or, more accurately, the relation between the volumes defined by the envelopes. The discussion may deal with multiple questions: Do these objects intersect? Is part of the partition common with adjacent objects? If so, the packing of objects may also be discussed, if it is sparse or dense.
2. The partition of space by a continuous and infinite surface which does not self-intersect. Each of these surfaces may divide the space into two subspaces which are present on either side of the surface. Such subspaces do not intersect or overlap.

3. n continuous, infinite surfaces which do not intersect or self-intersect, which divide the space into n+1 mutually exclusive subspaces.
4. An infinite yet non-continuous surface. That is, there exists openings within the surface which allow transition from one side of the surface to the other. Such a surface does not divide space.

2.2 The nature and definition of the phenomenon, and methods of understanding it

The research discusses the phenomenon of infinite continuous smooth minimal surfaces which partition space into two subspaces. Among all surfaces which partition space into two subspaces there exists a unique family of minimal surfaces which partition space into two identical subspaces.

The morphological discussion of the unique phenomenon of minimal surfaces which divide space into two identical subspaces allows focusing on the exploration of this specific phenomenon, studying it and making conclusions about the general phenomenon of partitioning space.

The purpose of the research is to deal with several subjects:

a. Investigating the phenomenon of dual identical periodic spaces and the partition between them.
b. The relation between order, organization of space, and its partition, to the symmetry groups which define this order.
c. The spectrum of possibility for the phenomenon. Is the number of surfaces fulfilling the requirements finite or infinite? Is there a way to sort them, topologically?

The phenomenon of partitioning space into two identical subspaces by smooth, continuous, repeating, minimal surfaces is a known phenomenon. Thus far, seven different surfaces which fulfill this requirement are known. The surfaces are shown in Figure 1, as a subsection of each.

The research points out the properties of these surfaces. Based on these properties, it develops tools to identify and discover surfaces which fit these properties.

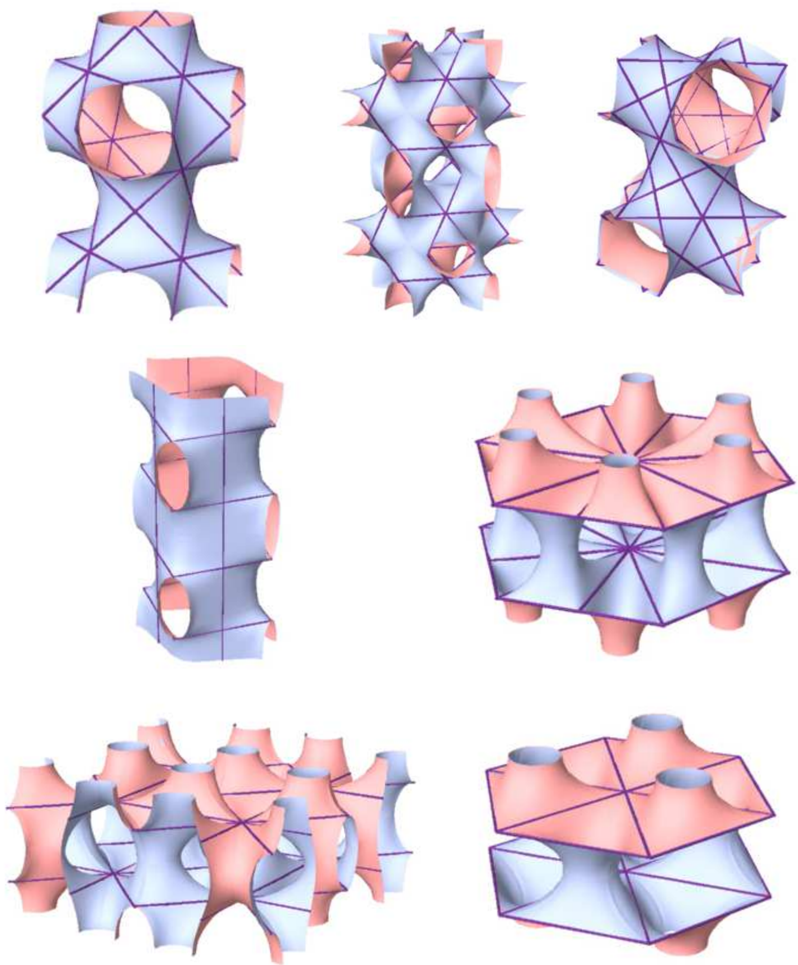


Figure 1 - Seven topologically different surfaces which divide the space into two identical subspaces.

3 Topological attributes of smooth surfaces which divide the space into two identical subspaces

3.1 Smooth surfaces – concepts and attributes

The discussed surfaces belong to the family of smooth two-dimensional surfaces. Smooth two-dimensional surfaces are characterized by the fact that all of their points are regular.

A regular point on the surface is a point in which all the tangents to the surface going through a single point are on a single plane. This plane is called the tangent plane to the surface. The line perpendicular to the tangent plane going through the same point is called the normal. Intersections between the surface and planes which contain the normal form curves which go through the regular point, and are called normal sections.

The tangent plane characterizes two types of surfaces:

a. A surface which is intersected by the tangent plane – a saddle-shaped surface (Figure 2).
b. A surface which is locally all on one side of the plane – a dome or cylindrical surface.

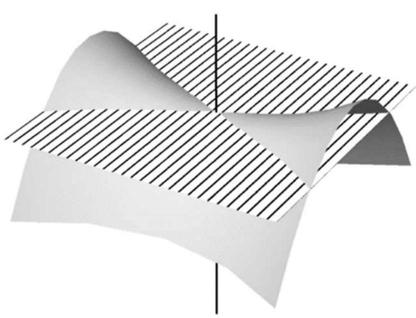


Figure 2 - Tangent plane to a saddle surface.

Amongst all the normal sections going through a regular point on a smooth surface, there exist two unique normal sections. One section represents a curve with maximal curvature. Another section represents a curve with minimal curvature. Both curves are called the principal curvatures of the surface, and are perpendicular to each other.

The average of the principal curvatures is called the “mean curvature”, while the product of the main curvature is called the “Gaussian curvature”.

The Gaussian curvature is a number which characterizes the surface and is a tool for topologically categorizing different surfaces (Figure 3) as follows:

a. A negative Gaussian curvature – characterizes saddle surfaces.
b. A zero Gaussian curvature – characterizes cylindrical surfaces.
c. A positive Gaussian curvature – characterizes dome surfaces.

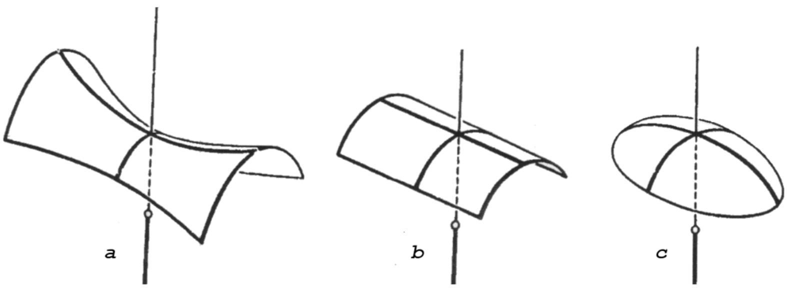


Figure 3 - Surfaces with different Gaussian curves.

3.2 Minimal surfaces which divide the space into two identical subspaces

From researching the phenomenon of partitioning space into two identical subspaces by the aforementioned surfaces, we learn about their characteristic topological properties, which are:

a. The surfaces are two-dimensional, smooth, saddle-shaped, and embedded in the three-dimensional Euclidean space.

b. The surfaces partition space into two identical subspaces. The two subspaces are often called dual spaces or complementary spaces or congruent spaces. The two spaces form two tunnels which intertwine and do not intersect. The two tunnels are identical in volume and shape, and their axes form two identical dual three-dimensional networks (Figure 4).

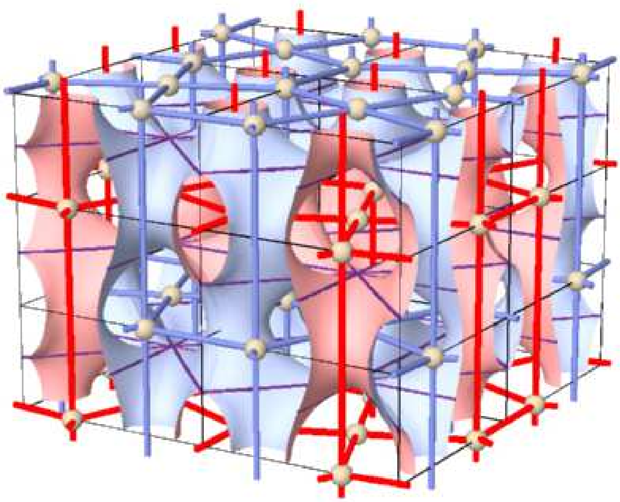


Figure 4 - Two dual tunnel networks and the surface buffering them.

The duality of these networks manifests in that a vertex of one interchanges with the volume of a "packing cell" of the dual network. Each three-dimensional, periodic network through space can be represented by tight packing (without spaces) of an object or multiple three-dimensional objects. The objects may have planar faces, smooth saddle-shaped faces, or a combination of both. These objects are called the "packing cells of the network". The vertices of one network are at the centers of the packing cells of its dual network. The edges of the networks, which connect its vertices, intersect the faces that are common to two adjacent packing cells. Each periodic network in space has exactly one dual network. Both networks are dual (Figure 5).

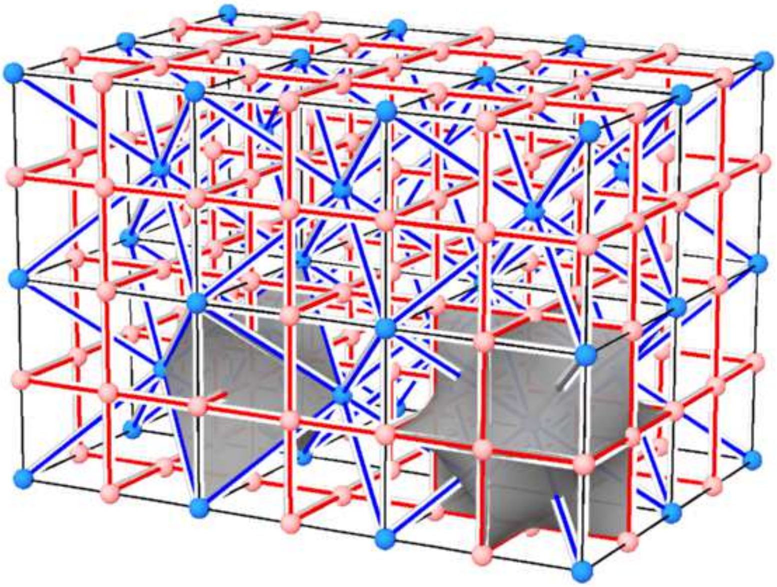


Figure 5 - Two dual networks and their packing units.

c. It is possible to transform one subspace to its dual via 180^0 rotation. The axes around which such a transformation is possible are embedded in the partitioning surface, and form a network which we call “2-fold axis network”. All edges of the network going through a single vertex (the intersection point of the rotation axes) are on a single plane.
d. The periodic nature of the surface indicates that the surface is made of basic periodic units. It also indicates the relation between the surface and the symmetry groups in three-dimensional space.

   We observe two forms of elementary periodic units of a surface which partitions space into two identical subspaces: One – A periodic unit bounded by a three-dimensional polygon formed by the 2-fold axes. The other – An elementary periodic unit that is derived by the symmetry elements that operate on the surface (Figure 6).

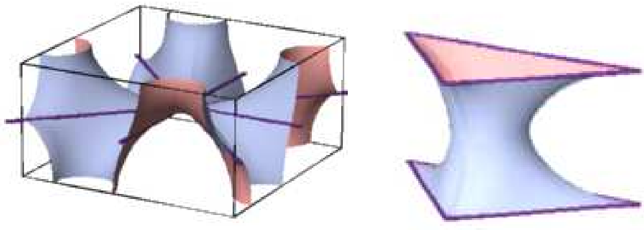


Figure 6 - Elementary periodic units.

The elementary periodic unit contains all of the properties of the surface, including representation of the 2-fold axes which rotate it into itself. The elementary periodic unit that is derived from the symmetry elements operating on the surface is represented within an elementary periodic region (E.P.R.) which will be farther discussed in the following chapter.

The E.P.R. embodies within it every periodic element belonging to the space it represents. In the case of the aforementioned surfaces, the surface is represented by an elementary periodic unit that is derived from the symmetry elements operating on the surface. This unit divides the volume of the periodic region into two identical volumes.

The network of 2-fold axes are represented by an axis or axes which go through the elementary periodic region and rotate it into itself. There are also cases in which the 2-fold axes network is represented by an axis or axes which rotate the elementary periodic region into an adjacent region. (Figure 7).

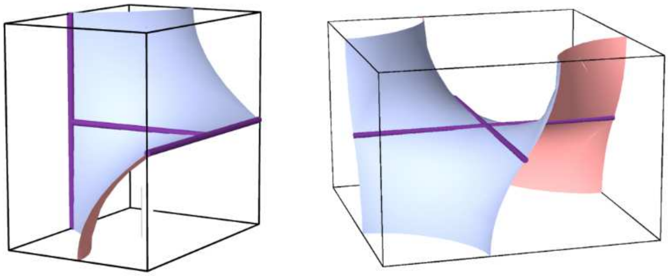


Figure 7 - 2-fold axes represented in the E.P.R.

e. The Euler genus of the periodic unit is an integer representing the number of cross-caps required to close the unit (Figure 8). By approximating the surface within the translation cell using a polyhedron, it is possible to calculate the genus g of the translation cell via the Euler characteristic, $F - E + V = 2(1 - g)$. Since Euler characteristic is invariant under subdivision, the calculation of the genus does not depend on the accuracy of the approximation.

   The Gaussian curvature can likewise be represented as a number dependent on the genus $K = K_{max} \times K_{min} = 4\pi(1\text{-}g)$

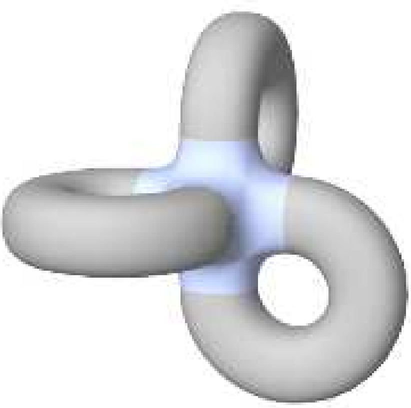


Figure 8 - Typical translation cell of a periodic surface with genus 3.

4 The method for searching and classifying the surfaces

4.1 The link between the surface attributes and symmetry groups

The properties of the desired surfaces, as previously described, resulted in the development of a method to search for and classify the surfaces.

The surfaces are defined as periodic, that is, made from a periodic unit. This property links the surfaces with the symmetry groups which operate in three-dimensional space.

The symmetry group is a combination of symmetry operations which operate on a point in space. By choosing points in space such that the environment around each point is identical, we form an infinite repeating lattice of points.

The number of different combinations of symmetry operations in three-dimensional space is finite. There are 230 different combinations, which are known as the 230 symmetry groups in three-dimensional space. Each of the 230 symmetry groups define a different point network. These networks are known as “crystallographic networks”. The crystallographic network is a platform for any regular periodic array.

“Periodic” means the existence of an elementary unit cell which cannot be farther divided by symmetry operations. The elementary periodic region (E.P.R.) represents, within it, all the properties of the periodic space.

An elementary periodic region which represents a region from one of the aforementioned surfaces will contain an elementary periodic unit which represents that surface.

4.2 Locating the Elementary Periodic Regions (E.P.R.) containing a piece of surface representing one of the required surfaces

An elementary periodic unit of a surface which partitions space into two identical subspaces, divides the volume of the E.P.R. into two identical complementary volumes. (Figure 9).

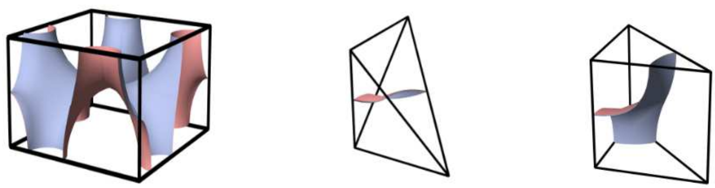


Figure 9 - Typical E.P.R.s and the periodic surface which divides them into two identical volumes.

An elementary periodic unit which is inside the E.P.R. contains a symmetry operation which transforms one volume into its identical complementary volume. This symmetry operation, as mentioned, is a 2-fold axis or axes (180^0 rotation) which go through the E.P.R. and rotate it into itself.

The E.P.R. which contain a 2-fold axis or axes may contain an elementary periodic unit of the surface. (Figure 10).

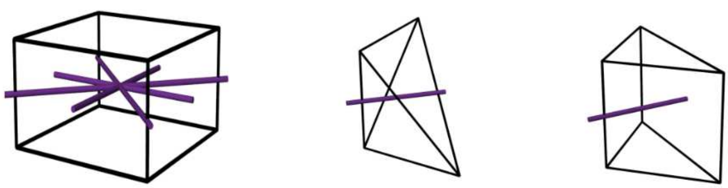


Figure 10 - Typical E.P.R.s which contain 2-fold axes.

4.3 Locating Elementary Periodic Unit of the surface

Replicating the E.P.R, which contains 2-fold axis or axes which rotate it into itself, leads to the creation of a 2-fold axis network (Figure 11).

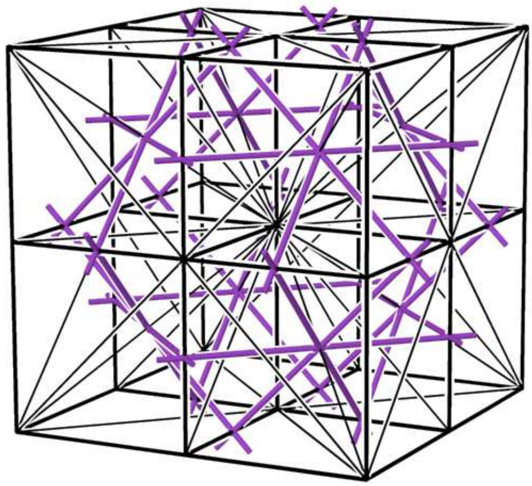


Figure 11 - 2-fold axis network.

The 2-fold axis network is a periodic network, meaning, it was created by the replication of a periodic segment. In these networks, it is possible to find a large number of different periodic units, from the basic network segment which is bounded by the E.P.R. and up to a cell which can be replicated by translation alone. We say that the size of the periodic unit depends on the degree of periodicity of the unit. The most elementary periodic unit (contained in the E.P.R.) is a unit with the highest periodicity, whereas groups of said unit, up to the size of a translation unit, are of a lower degree of periodicity.

Amongst all the periodic units of the 2-fold axis network, there exists a unit with a bounded periodic cell made of 2-fold axes segments (Figure 12).

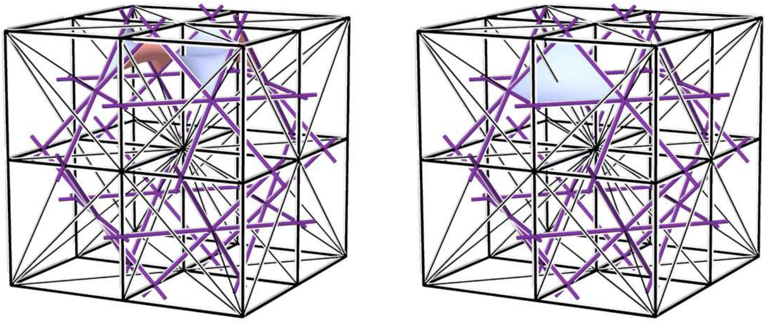


Figure 12 - Periodic cells and the minimal surface segment bounded by them.

In exploring the properties of the smooth surfaces which partition space into two identical subspaces, we have come to see that the periodic units of the 2-fold axis network are a boundary of the periodic units of the surface. Meaning, a periodic unit of the surface is bounded in the periodic cell of the 2-fold axis network.

Locating the periodic cell within the 2-fold axis network, and the surface segment bounded by it, and replicating it through 2-fold rotation operations, would lead to the creation of a smooth periodic surface which partitions space into two identical subspaces (Figure 13).

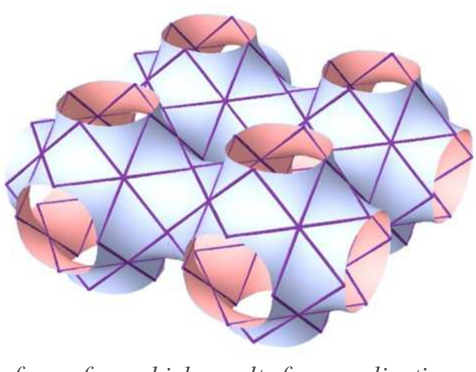


Figure 13 - Part of a surface which results from replicating a periodic cell containing a periodic surface segment.

We go farther to say that any motif bounded by the periodic cell of a 2-fold axis network would lead to a periodic surface which partitions space into two identical subspaces.

Possible motifs are:

a. The minimal surface bounded by the cell
b. A polyhydric segment resulting in the creation of an infinite polyhedron.
c. A surface segment made from planar faces, which may lead to the creation of an infinite polyhedron which may or may not be regular.
d. Any surface segment that is regular or has singularity points (Figure 14).

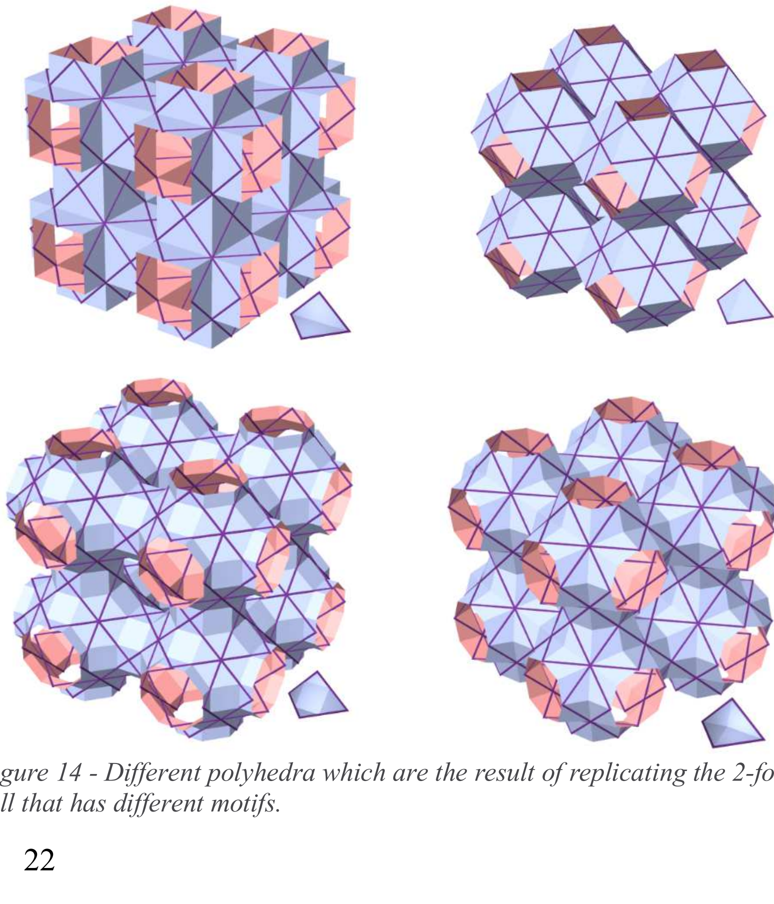


Figure 14 - Different polyhedra which are the result of replicating the 2-fold cell that has different motifs.

4.4 Topological classification of the surfaces

The method for locating the surfaces which divide space into two identical subspaces yields many surfaces which are topologically similar.

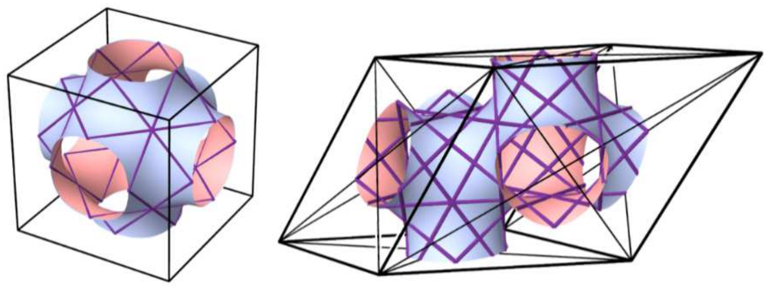


Figure 15 - Typical translation cells of surfaces which divide space into two identical subspaces.

There are two physical characteristics to the aforementioned surfaces. One is the shape of the translation cell of the surface. The other is the geometrical shape of the two subspaces on either side of the surface (Figure 15*)*.

The surface partitions space into two identical mazelike subspaces made from two continuous intertwined tunnels. The axes of the tunnels form a periodic continuous network which is called “the tunnel network”. The relation between the two networks is dual-complementary, and each can be defined by the other (Figure 16).

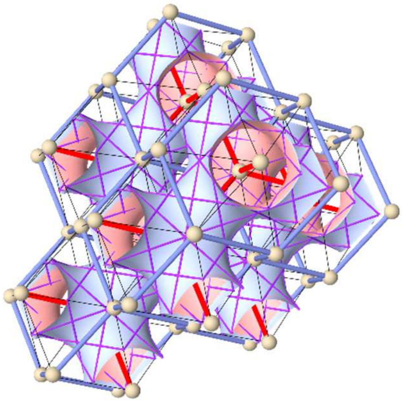


Figure 16 - Dual-complementary tunnel networks and the surface separating them.

Both of the aforementioned characteristics derive from one another. Replicating the translation cell leads to the creation of the surface which divides space into two tunnel-like subspaces, vice versa.

There is a one-to-one relation between a typical translation cell and the tunnel network. Surfaces which may appear different, as well as elementary surface segments which are bounded by different E.P.R.s may be characterized by similar tunnel networks and lead to a similar translation cell. Surfaces characterized by a similar tunnel network are topologically similar.

Categorizing the surfaces based on the tunnel networks or typical translation cell, may lead to the location of surfaces which are topologically different.

4.5 Summary of the method for locating and classifying the surfaces

The method for locating the surfaces is made from sequential phases, each of which brings us closer to the target, and narrows down the search area.

Performing each of the steps of the method leads to the location of all smooth surfaces which divide space into two identical subspaces. Smooth surfaces are surfaces which their surface segment, bounded by the cell bounded by the 2-fold axis network is the minimal surface. Amongst the surfaces found there will be surfaces which may appear different, but are topologically identical.

The phases of the method are as follows:

a. Locating the elementary regions defined by the symmetry groups which operate in the Euclidean space.
b. Filtering down to the elementary regions which contain 2-fold axes which rotate them into themselves. Meaning, they may contain a surface segment which divides their volume into two identical complementary volumes.
c. Locating 2-fold axis networks formed by replicating the located elementary regions.
d. Locating a periodic cell bounded by the 2-fold axis network, and finding the elementary surface segment bounded by it.
e. Replicating the bounded cell until a large enough segment of the surface, which allows identifying the tunnel network or a typical translation cell, emerges.
f. Defining the dual-complementary tunnel networks.
g. Presenting the E.P.R which contains the representation of the 2-fold axes, the surface segment, and both of the dual networks.
h. Categorizing the surfaces based on the distinct dual tunnel networks.

5 The process of searching for and classifying the surfaces

5.1 Locating the Elementary Periodic Regions (E.P.R.s)

A periodic space is an array created by the replication of basic three-dimensional units, which are packed without leaving any spaces, that is, they cover the entire space.

The elementary unit is shaped as a three-dimensional box, which is called a “typical translation cell”. The repetition is expressed by translation of the cell in the three dimensions of space.

Different typical translation cells are characterized by the relations between the lengths of the edges of the cell, and the angle between them. The different combinations of possible ratios between the lengths of the edges of the cell and the angle between them, define only 7 typical translation cells.

The vertices of a typical translation cell, which are packed next to each other in a sequence, so that they cover the entire space, create an organized periodic array of points. The aforementioned system of points is called a crystallographic system (Figure 17).

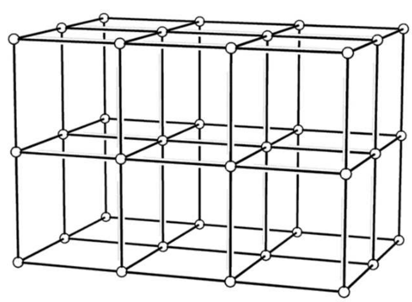


Figure 17 - Crystallographic system made by replicating a translation cell.

We will note that replicating the typical translation cell, to receive the entire space, is done by translation only. Translation is the most elementary symmetry operation. Translation cells may contain within them other symmetry operations besides translation. A symmetry operation which is applied to the array of points, around one of the points in the array, causes all of the points to move, other than the point around which the symmetry operation was applied, and take the position of other points in the array. The whole structure of the array of points has not changed its form after the operation.

The symmetry operations in three-dimensional space are:

a. Mirror planes

b. 2-fold (180^0), 3-fold (120^0), 4-fold (90^0), or 6-fold (60^0) rotation axes.

c. Inversion points and axes.

A group of different combinations of symmetry operations acting around a point is called a symmetry group. When we examine each of the seven different systems, we discover that there is an essential symmetry element. The essential symmetry element is joined by additional symmetry elements up to the maximal different possible combinations. This class is called the holosymmetric class. All of the combinations on the path to the holosymmetric class are called symmetry sub-classes of the system. The sum of all possible combinations, meaning, all symmetry classes in three-dimensional space, comes up to 230 classes.

The seven different systems are:

a. Triclinic system

Ratio between edges $\mathbf{a \neq b \neq c}$

Angle ratio $\alpha \neq \beta \neq \gamma \neq 90°$

Symmetry elements
Center inversion point

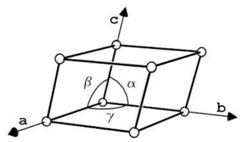


Figure 18 - A translation cell of the Triclinic system

b. Monoclinic system

Ratio between edges $\mathbf{a \neq b \neq c}$

Angle ratio $\alpha = \gamma = 90°, \beta \neq 90°$

Symmetry elements
Center inversion point
1 mirror plane
1 2-fold axis (Essential)

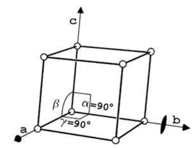


Figure 19 - A translation cell of the Monoclinic system

c. Orthorhombic system

Ratio between edges $\mathbf{a \neq b \neq c}$

Angle ratio $\alpha = \beta = \gamma = 90°$

Symmetry elements
Center inversion point
3 mirror planes
3 2-fold axes (Essential)

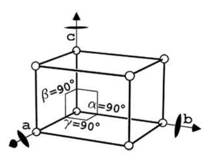


Figure 20 - A translation cell of the Orthorhombic system

d. Tetragonal system

Ratio between edges $\mathbf{a} = \mathbf{b} \neq \mathbf{c}$

Angle ratio $\alpha = \beta = \gamma = 90^\circ$

Symmetry elements
Center inversion point
5 mirror planes
4 2-fold axes
1 4-fold axis (Essential)

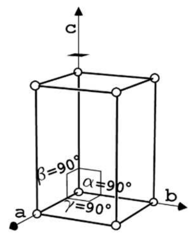


Figure 21 - A translation cell of the Tetragonal system

e. Cubic system

Ratio between edges $\mathbf{a} = \mathbf{b} = \mathbf{c}$

Angle ratio $\alpha = \beta = \gamma = 90^\circ$

Symmetry elements
Center inversion point
9 mirror planes
6 2-fold axes
3 4-fold axes
4 3-fold axes (Essential)

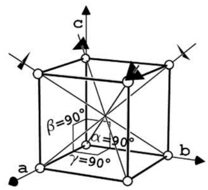


Figure 22 - A translation cell of the Cubic system

f. Trigonal system

Ratio between edges $\mathbf{a} = \mathbf{b} = \mathbf{c}$

Angle ratio $\alpha = \beta = \gamma < 120^\circ \neq 90^\circ$

Symmetry elements
Center inversion point
3 mirror planes
3 2-fold axes
1 3-fold axis (Essential)

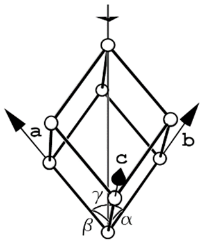


Figure 23 - A translation cell of the Trigonal system

g. Hexagonal system

Ratio between edges $a = b \neq c$

Angle ratio $\alpha = \beta = 90°, \gamma = 120°$

Symmetry elements
Center inversion point
7 mirror planes
6 2-fold axes
1 6-fold axis (Essential)

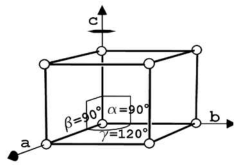


Figure 24 - A translation cell of the Hexagonal system

The tight packing of each of the seven typical translation cells forms a different crystallographic network. The vertices of the translation cells which form the points of the network are the location around which the symmetry groups operate. When we examine the aforementioned seven networks, we find that there are other possible points in the network space in which the symmetry groups may be placed. That is, the symmetry groups operate on the network around that point without altering the network.

These points join the whole array and create additional crystallographic networks. The location of the additional points in space is not random. Each new point has a representation in the typical translation cell.

The location of the point on the cell appears in four different forms (Figure 25):

a. At the vertices of the cell – A primitive cell, marked as P
b. At the centers of the bases of the cell and its vertices – A base centered cell, marked by C.
c. At the centers of the faces of the cell and its vertices – A face centered cell, marked by F.

d. At the center of the cell and its vertices – A body centered cell, marked by I.

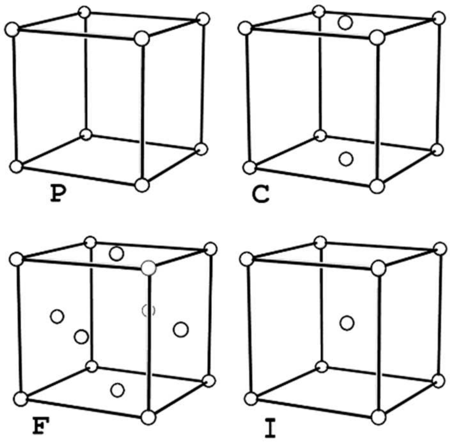


Figure 25 - Four different types of cells based on the appearance of symmetry points

P – primitive cell

C – base centered

F – face centered

I – body centered

The position of the additional points arises from the geometric shape of the cells. That is, it is dependent on the relation of the lengths of the edges, and the relation of the angles between them.

A primitive cell exists in all seven systems.

A base centered cell exists in the triclinic and orthorhombic systems.

A face centered cell exists in the orthorhombic and cubic systems.

A body centered cell exists in the orthorhombic, tetragonal and cubic systems.

The different aforementioned combinations create 14 crystallographic networks which are called "Bravais lattices"[1] (Figure 26).

[1] Auguste Bravais, 1811 – 1863, a French physicist known for his work in crystallography, the conception of Bravais lattices, and the formulation of Bravais law.

The 14 translation cells which represent the 14 Bravais lattices, may be farther divided by the symmetry elements which exist amongst the points of the cell. The smallest cell, which cannot be divided farther is called the Elementary Periodic Region (E.P.R.) of the network.

The vertices, edges, and faces of the E.P.R., which is derived from the typical translation cell, are the points of intersection between the symmetry elements of the class of the translation cell.

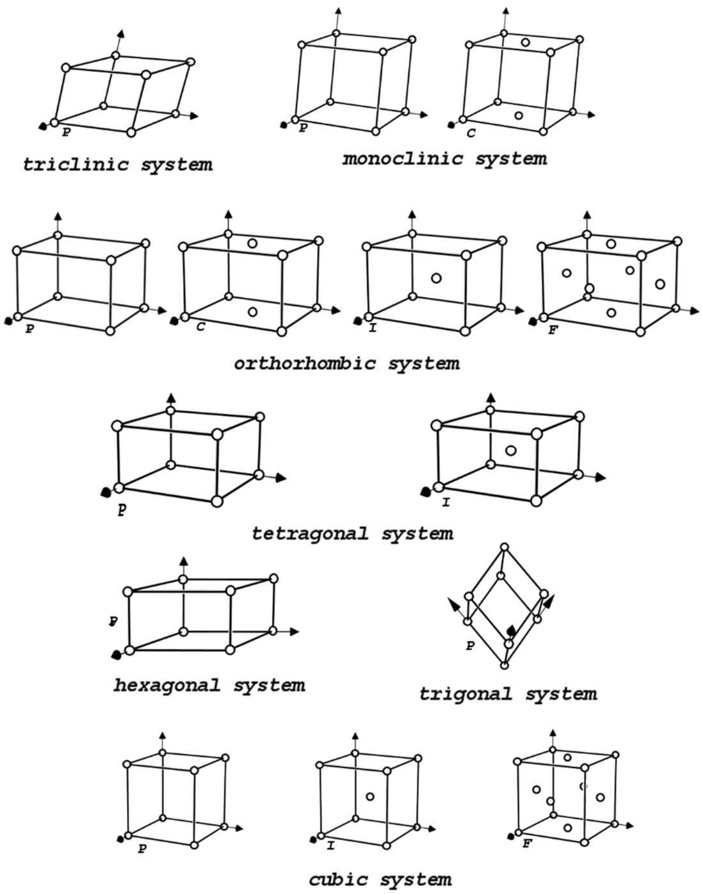


Figure 26 - The 14 Bravais lattices

The E.P.R. is a representation of the entire space, and contains within it a representation for each periodic element which is contained in the space, including, first and foremost, the symmetry group.

The edges of the E.P.R. may represent rotation axes of the symmetry group, or a place of intersection between two mirror planes. The faces may be mirror planes of the group.

The E.P.R. is generally part of the translation cell, but may be equal in size to the translation cell, or even be derived from multiple adjacent translation cells.

The E.P.R. appears in three different geometric shapes (Figure 27):

a. A tetrahedral structure
b. A prism with a rectangular base
c. A prism with a triangular base

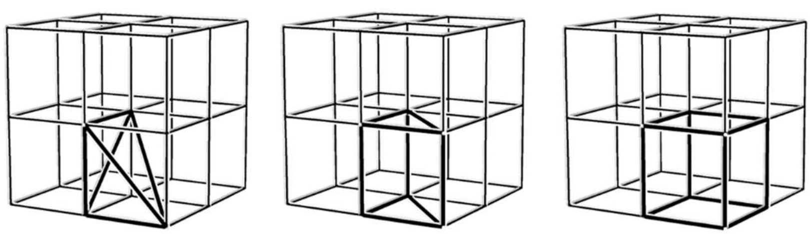


Figure 27 - Typical E.P.R.s

The inverse operation to deriving the E.P.R. is its replication. That is, reconstructing the space by replicating the E.P.R. using the symmetry elements.

Locating the E.P.R.s which represent the different symmetry groups can be done by dividing the 14 translation cells which represent the 14 Bravais lattices using the symmetry elements of the holosymmetric symmetry group, which contains the maximal number of possible symmetry elements. This will derive the smallest possible elementary cell. A sub-group of the holosymmetric group will derive a larger cell, which is formed by a replication of the smallest cell.

a. Triclinic system

The translation cell of the triclinic system does not contain additional symmetry elements besides translation. The cell can therefore not be divided farther. The E.P.R. of the triclinic system is identical in size to the typical translation cell (Figure 28).

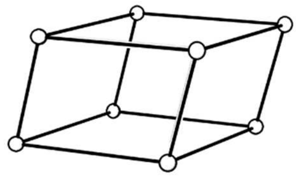


Figure 28 - E.P.R. of the primitive cell of the triclinic system

b. Monoclinic system

A typical translation cell of the monoclinic system that is not farther divided by its symmetry elements. The E.P.R. of the symmetry group represented by this cell is therefore identical in size to the typical translation cell (Figure 29).

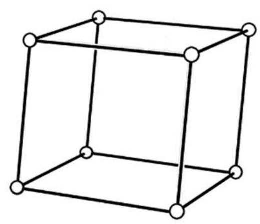


Figure 29 - E.P.R. of the primitive cell of the monoclinic system

The E.P.R. which is formed by dividing a base centered translation cell by the elements of the associated symmetry group. The E.P.R. is a quarter of the volume of the translation cell (Figure 30).

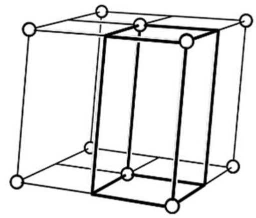


Figure 30 - E.P.R. of the base centered cell of the monoclinic system

c. Orthorhombic system

A typical translation cell of the orthorhombic system that is not farther divided by its symmetry elements. The E.P.R. of the symmetry group represented by this cell is therefore identical in size to the typical translation cell (Figure 31).

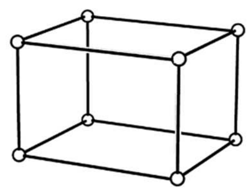


Figure 31 - E.P.R. of the primitive cell of the orthorhombic system

The E.P.R. which is formed by dividing a base centered translation cell by the elements of the associated symmetry group. The E.P.R. is a quarter of the volume of the translation cell (Figure 32).

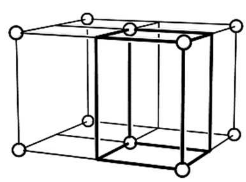


Figure 32 - E.P.R. of the base centered cell of the orthorhombic system

The E.P.R. which is formed by dividing a face centered translation cell by the elements of the associated symmetry group. The E.P.R. is one eighth of the volume of the translation cell (Figure 33).

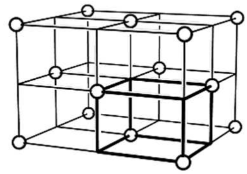


Figure 33 - E.P.R. of the face centered cell of the orthorhombic system

The E.P.R. which is formed by dividing a body centered translation cell by the elements of the associated symmetry group. The E.P.R. is one eighth of the volume of the translation cell (Figure 34).

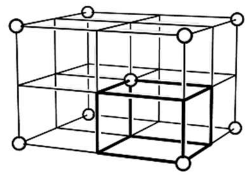


Figure 34 - E.P.R. of the body centered cell of the orthorhombic system

d. Tetragonal system

A typical translation cell of a tetragonal system that does not contain mirror planes and that is not farther divided by its symmetry elements. The E.P.R. of the symmetry group represented by this cell is therefore identical in size to the typical translation cell (Figure 35).

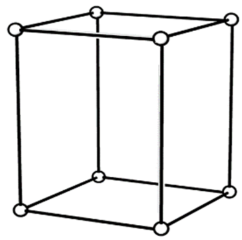


Figure 35 - E.P.R. of the primitive cell of the tetragonal system not including mirror operations

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the associated symmetry group which contains mirror planes. The E.P.R. is a quarter of the volume of the translation cell (Figure 36).

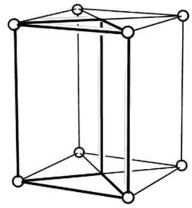


Figure 36 - E.P.R. of the primitive cell of the tetragonal system including mirror operations

The E.P.R. which is formed by dividing a body centered translation cell by the elements of the associated symmetry group which does not contain mirror planes. The E.P.R. is one eighth of the volume of the translation cell (Figure 37).

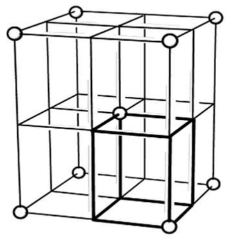


Figure 37 - E.P.R. of the body centered cell of the tetragonal system not including mirror operations

The E.P.R. which is formed by dividing a body centered translation cell by the elements of the associated symmetry group which contains mirror planes. The E.P.R. is one sixteenth of the volume of the translation cell (Figure 38).

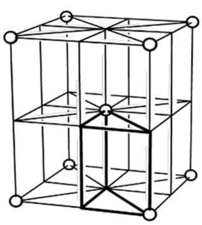


Figure 38 - E.P.R. of the body centered cell of the tetragonal system including mirror operations

e. Cubic system

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the holosymmetric group. The E.P.R. is one part in twenty-four of the volume of the translation cell (Figure 39).

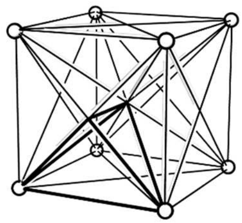


Figure 39 - E.P.R. of the primitive cell of the cubic system

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the symmetry group which does not contain mirror planes. The E.P.R. is one sixth of the volume of the translation cell (Figure 40).

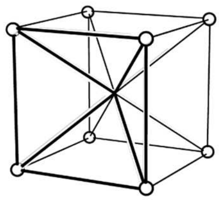


Figure 40 - E.P.R. of the primitive cell of the cubic system not including mirror operations

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the symmetry group which does not contain mirror planes or 4-fold rotation symmetry. The E.P.R. is one twelfth of the volume of the translation cell (Figure 41).

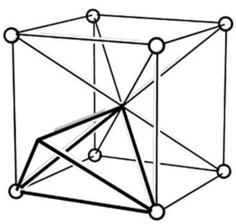


Figure 41 - E.P.R. of the primitive cell of the cubic system not including mirror operations or 4-fold rotation axes

The E.P.R. which is formed by dividing a body centered translation cell by the elements of the holosymmetric group. The E.P.R. is one part in forty-eight of the volume of the translation cell (Figure 42).

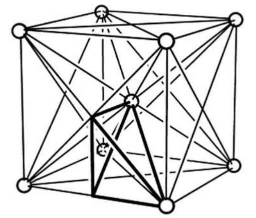


Figure 42 - E.P.R. of the body centered cell of the cubic system

The E.P.R. which is formed by dividing a body centered translation cell by the elements of the symmetry group which does not contain mirror planes or 4-fold rotation symmetry. The E.P.R. is one twelfth of the volume of the translation cell (Figure 43).

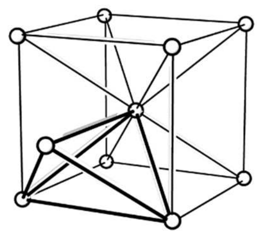


Figure 43 - E.P.R. of the body centered cell of the cubic system not including mirror operations or 4-fold rotation axes

The E.P.R. which is formed by dividing a face centered translation cell by the elements of the associated symmetry group. The E.P.R. is one part in ninety-two of the volume of the translation cell (Figure 44).

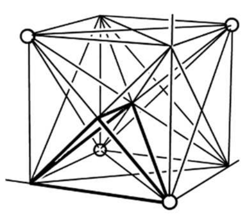


Figure 44 - E.P.R. of the face centered cell of the cubic system

f. Trigonal system

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the holosymmetric group. The E.P.R. is one sixth of the volume of the translation cell (Figure 45).

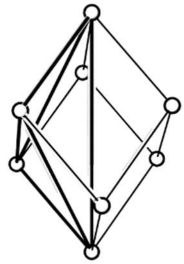


Figure 45 - E.P.R. of the primitive cell of the trigonal system

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the symmetry group which does not contain mirror planes. The E.P.R. is one third of the volume of the translation cell (Figure 46).

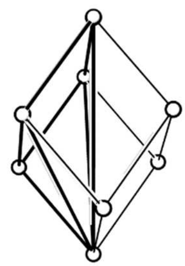


Figure 46 - E.P.R. of the primitive cell of the trigonal system not including mirror operations

g. Hexagonal system

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the holosymmetric group. The E.P.R. is one twelfth of the volume of the translation cell (Figure 47).

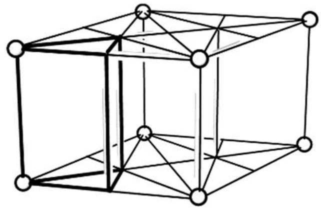


Figure 47 - E.P.R. of the primitive cell of the hexagonal system

The E.P.R. which is formed by dividing a primitive translation cell by the elements of the symmetry group which does not contain mirror planes. The E.P.R. is one half of the volume of the translation cell (Figure 48).

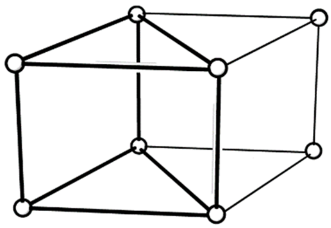


Figure 48 - E.P.R. of the primitive cell of the hexagonal system not including mirror operations

The translation cell of the hexagonal system does not contain additional symmetry elements besides translation. The cell can therefore not be divided farther. The E.P.R. of the triclinic system is identical in size to the typical translation cell (Figure 49).

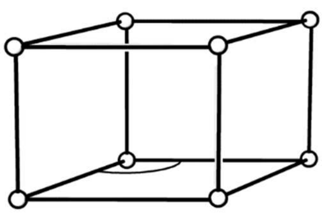


Figure 49 - E.P.R. of the primitive cell of the hexagonal system including only translations

5.2 Filtering the Elementary Periodic Regions (E.P.R.s) which contain 2-fold axes which rotate them into themselves

As mentioned in the chapter dealing with the method for locating the surfaces, an E.P.R. which may contain a representation of a surface which partitions space into two identical subspaces will also contain the operation which rotates one subspace into its dual-complementary subspace. This operation is a 2-fold axis or axes which are represented within the E.P.R. and rotate it into itself.

The process of filtering the E.P.R.s found in the previous chapter down to the ones which contain a 2-fold axis or axes is done systematically, using the rules derived from examining the known examples of surfaces which partition space into two identical subspaces.

A matching 2-fold axis must follow the following rules:

a. The axis passes through the center of the E.P.R.
b. If the axis intersects the edges of the E.P.R., the intersection would be in the middle point of the edge
c. If the axis goes through the face of the E.P.R., the intersection point will be at the center of the face
d. The axis may pass through opposite edge middle points
e. The axis may pass through opposite face centers
f. The axis may pass through the center of an edge and the center of a face that is opposite from that edge
g. If the E.P.R. contains more than two axes, they will all be on a single plane. In case an E.P.R. contains several axes, which are not on the same plane, all the different planes which contain the maximal number of axes must be located.

On the exterior of the E.P.R.s which contain 2-fold axes which rotate them into themselves, we will locate 2-fold axes which rotate them into a neighboring E.P.R. while maintaining the above rules.

Filtering down to the E.P.R.s which may contain a 2-fold axis or axes, out of the sum total of E.P.R.s located in the previous chapter, will be done individually for each system. We will mark each E.P.R. using the shortened system name and a serial number, based on the order of locating it.

a. The Triclinic system

The E.P.R. is equal in size to the typical translation cell and has the lowest degree of symmetry. It does not contain any 2-fold axes which rotate it into itself (Figure 50).

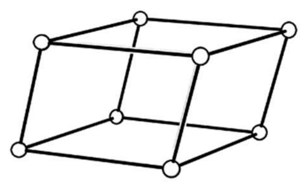


Figure 50 - E.P.R. of the primitive cell of the Triclinic system

b. The Monoclinic system

A primitive cell of the Monoclinic system does not contain 2-fold axes which rotate it into itself (Figure 51).

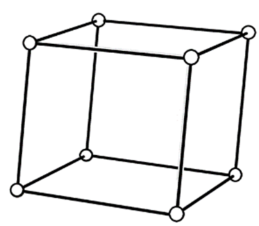


Figure 51 - E.P.R. of the primitive cell of the Monoclinic system

The E.P.R. which is derived from a base centered cell does not contain 2-fold axes which rotate it into itself (Figure 52).

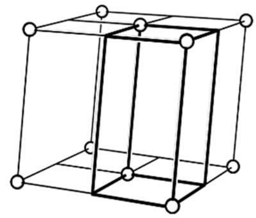


Figure 52 - E.P.R. of the base centered cell of the Monoclinic system

c. The Orthorhombic system

The E.P.R.s derived from all of the cell types of the Orthorhombic system are in the shape of boxes with the same ratios between their edge lengths and angles. They are all similar to the translation cell (Figure 53).

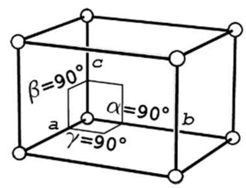
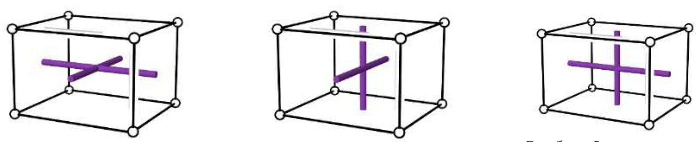


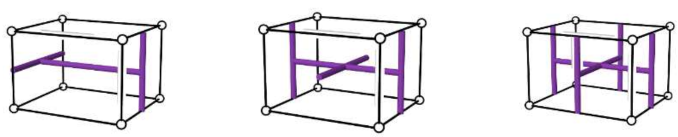
Figure 53 - E.P.R. of the Orthorhombic system

Locating the different options for passing 2-fold axes inside the E.P.R. (Figure 54) and on its exterior (Figure 55)



Ortho.1 *Ortho.2* *Ortho.3*

Figure 54 - The different options for passing 2-fold axes through the Orthorhombic system



Ortho.4 *Ortho.5* *Ortho.6*

Figure 55 - The different options for passing 2-fold axes on the exterior of the Orthorhombic system

d. The Tetragonal system

The E.P.R.s derived by the symmetry group of the Tetragonal system take two forms. One form is a prism of which base is a right-angle isosceles triangle. The other is that of a prism with a rectangular base (Figure 56).

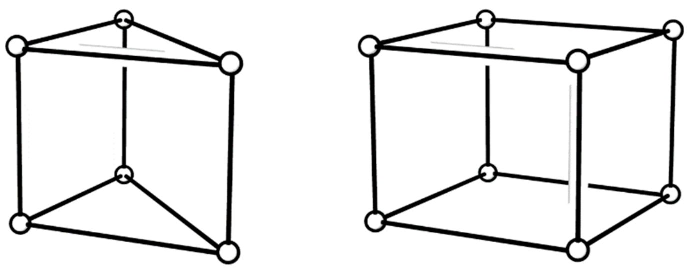
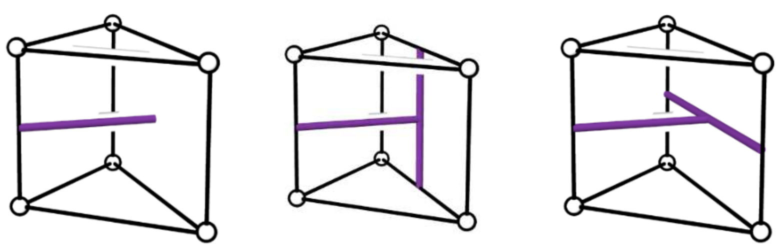


Figure 56 - E.P.R.s of the Tetragonal system

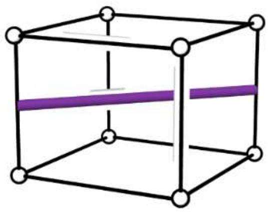
Locating the different options for passing 2-fold axes through and on the exterior of the first form (Figure 57).



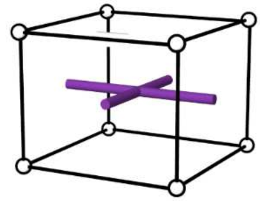
Tet. 1 *Tet. 2* *Tet. 3*

Figure 57 - Different ways to pass 2-fold axes through the triangular based prism shaped E.P.R. of the Tetragonal system

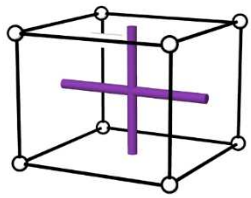
Locating the different options for passing 2-fold axes through and on the exterior of the second form (Figure 58).



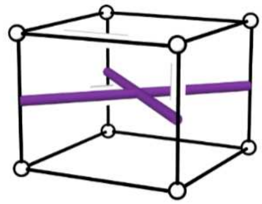
Tet. 4



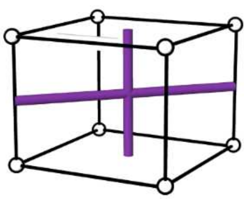
Tet. 5



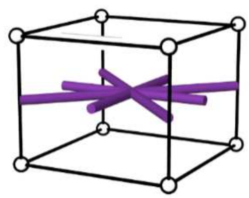
Tet. 6



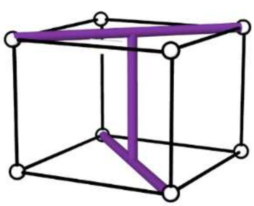
Tet. 7



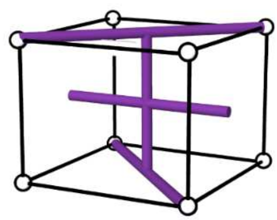
Tet. 8



Tet. 9



Tet. 10

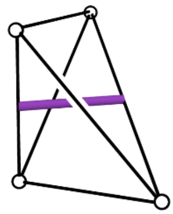


Tet. 11

Figure 58 - Different ways to pass 2-fold axes through the rectangular based prism shaped E.P.R. of the Tetragonal system

e. The Cubic system

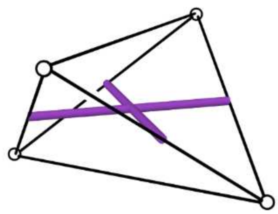
The E.P.R.s derived by the symmetry group of the Cubic system take two forms. One is the form of a tetrahedron, with a volume that is one part in forty-eight of the cube. The other, also a tetrahedron, with a volume that is one twelfth of the cube. The first form contains one option for passing a 2-fold axis which rotates it into itself (Figure 59).



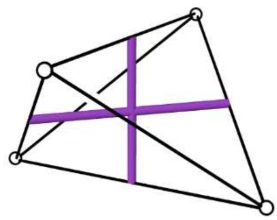
Cub. 1

Figure 59 - An E.P.R. of the Cubic system, and the 2-fold axis which can be passed through it

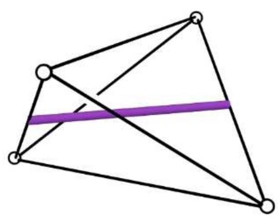
Locating the different options for passing 2-fold axes through and on the exterior of the second form (Figure 60).



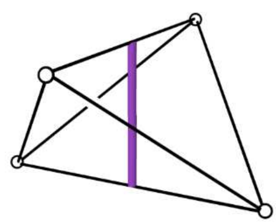
Cub. 2



Cub. 3



Cub. 4

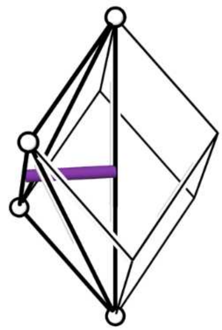


Cub. 5

Figure 60 - Different options for passing 2-fold axes through an E.P.R. of the Cubic system

f. The Trigonal system

The Trigonal system contains an E.P.R. that is shaped as a tetrahedron, and there is one option for passing a 2-fold axis which rotates it into itself (Figure 61).



Tri. 1

Figure 61 - An E.P.R. of the Trigonal system, and the 2-fold axis which can be passed through it

g. The Hexagonal system

The E.P.R.s of the Hexagonal system take two forms. One, in the form of a prism of which base is an equilateral triangle. The other, in the form of a box, in which the ratios between the lengths of the edges of the box rare equal to those of the translation cell (Figure 62).

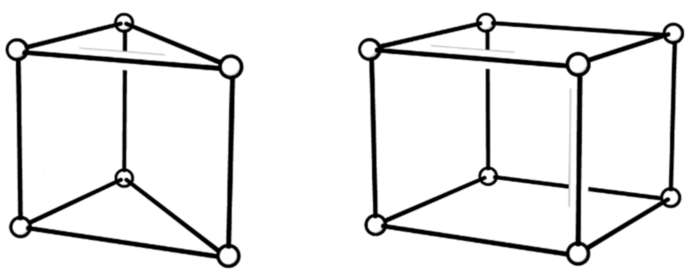
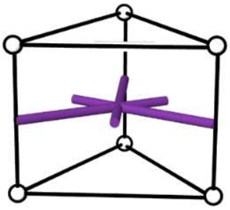
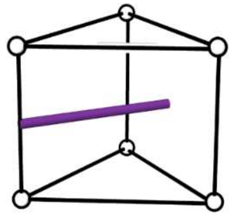


Figure 62 - E.P.R.s of the Hexagonal system

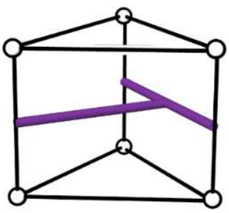
Locating the different options for passing 2-fold axes through and on the exterior of the first form (Figure 63).



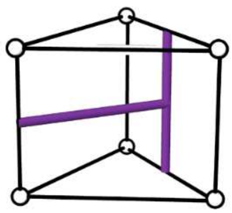
Hex. 1



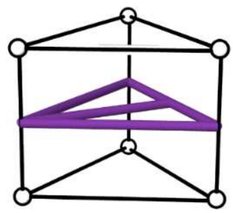
Hex. 2



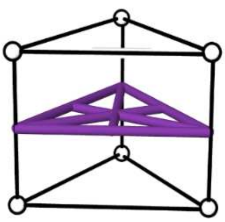
Hex. 3



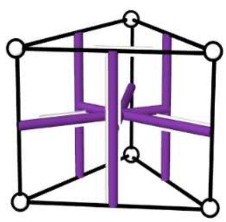
Hex. 4



Hex. 5



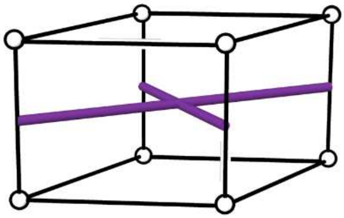
Hex. 6



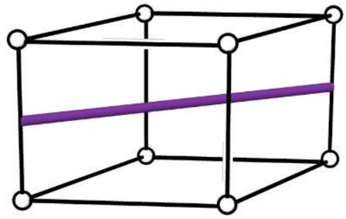
Hex. 7

Figure 63 - Different ways to pass 2-fold axes through the triangular based prism shaped E.P.R. of the Hexagonal system

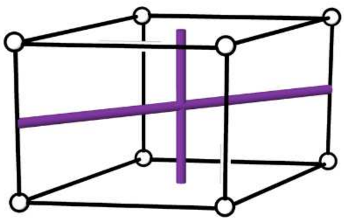
Locating the different options for passing 2-fold axes through and on the exterior of the second form (Figure 64).



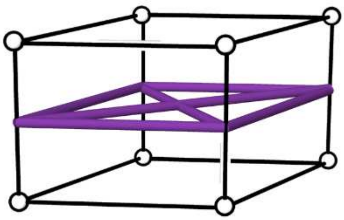
Hex. 8



Hex. 9



Hex. 10



Hex. 11

Figure 64 - Different ways to pass 2-fold axes through the rectangular based prism shaped E.P.R. of the Hexagonal system

5.3 The method for locating the 2-fold axis networks

Locating the 2-fold axis networks is done by replicating each of the E.P.R.s which contain a 2-fold axis or axes (As were determined in the previous chapter), until receiving a network segment in which the periodic cells which form the network can be located. The periodic cell is one which is bounded by the 2-fold axes of the network. These networks, called "2-fold axis networks”, are categorized topologically based on the periodic cells bounded by them.

It is possible that replicating different E.P.R.s, including ones created from different symmetry systems, may result in topologically identical 2-fold axis networks. That is, the periodic cells of the networks would be topologically identical.

In some cases, the 2-fold axis network may contain more than one periodic cell. That is, multiple periodic cells may be located, which form the same network.

The periodic cell of the 2-fold axis network may appear in one of two forms (Figure 65):

a. A connected three-dimensional polygon, which is derived from the network, and is a periodic unit in which a periodic segment of the surface is bounded.
b. Two identical parallel planar polygons, which are separate from each other. The surface segment bounded between them forms a sleeve around an axis that passes through the center of both polygons. We call this cell a “Split Polygon Cell”. A periodic split polygon cell is derived from a multi-layered 2-fold axis network.

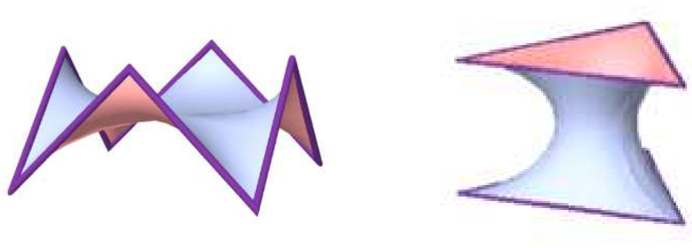


Figure 65 - Periodic cells and the surface segments bounded by them

The periodic cell which characterizes the different 2-fold axis networks will be shown, for demonstration purposes, along with an elementary surface segment that is bounded by it.

The shown surface segment is a minimal surface. A surface which may be formed by dipping the periodic cell in a soap solution.

Replicating the periodic cell along with the surface segment leads to the location of periodic surfaces which partition space into two identical subspaces. We observe that these surfaces are not unique, and refer to them as “soap solution surfaces” (That is, surfaces which form by dipping their closed cell in a soap solution).

We have already demonstrated that any motif which is enclosed by the periodic cell leads to the creation of a two-dimensional surface which partitions space into two identical subspaces. The question arises, then, whether different motifs enclosed in the same cell may lead to the creation of topologically different surfaces which partition space into two identical subspaces. This question will be farther discussed later on.

The 2-fold axis networks will be numbered by the order in which they were located. It is important to note that the number of different 2-fold axis networks is finite. The proof of this lies in the fact that the number of symmetry groups in three-dimensional space is also finite (230 groups). From this fact, we can derive that the number of E.P.R.s is also finite, and hence so are the number of E.P.R.s in which there is a 2-fold axis or axes. Replicating those E.P.R.s hence leads to a finite number of 2-fold axis networks.

5.4 The process of locating the 2-fold axis networks

The Triclinic and Monoclinic systems did not have any E.P.R.s containing a 2-fold axis or axes which rotate them into themselves, or a neighboring E.P.R.

In the Orthorhombic system, replicating the E.P.R.s Ortho.1, Ortho.2 and Ortho.3 (Figure 66) leads to the formation of a multi-layered 2-fold network. All of the layers are disconnected, parallel and at equal distances from each other. Each layer is a planar network of squares or rectangles (Figure 67). We call this network "Net1" (Network number 1).

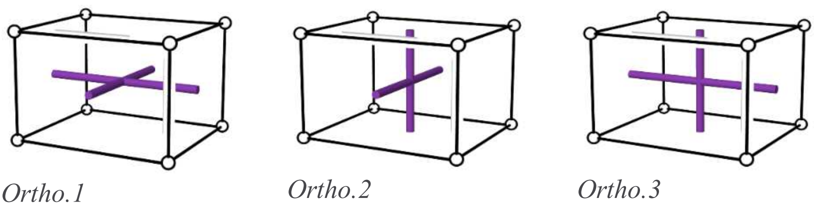


Figure 66 - The E.P.R.s Ortho.1, Ortho.2 and Ortho.3 along with their 2-fold axes

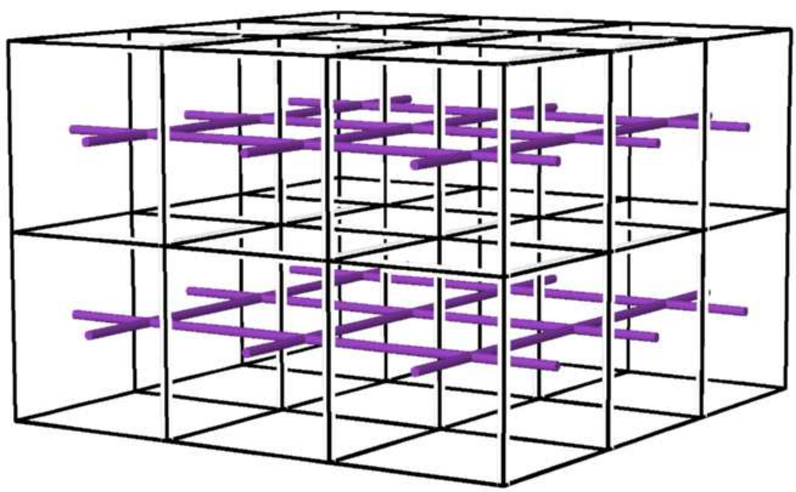


Figure 67 - The 2-fold axis network Net1

The periodic cell of Net1 is a split polygon cell with rectangular bases (Figure 68).

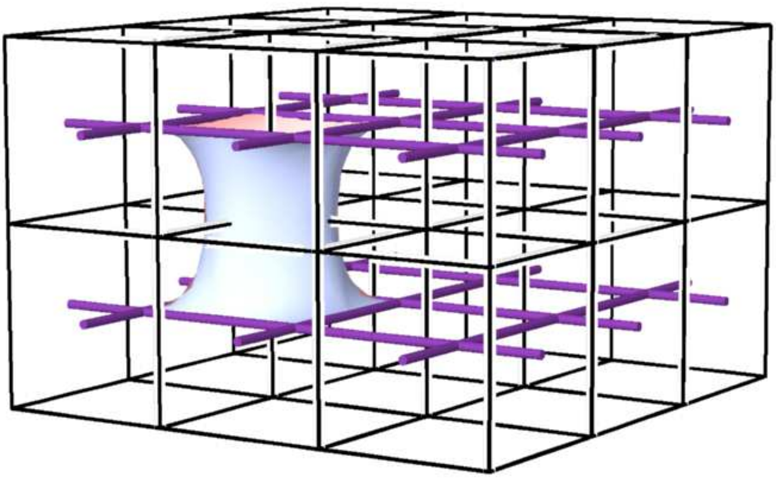
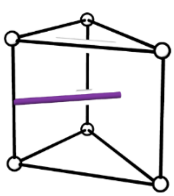
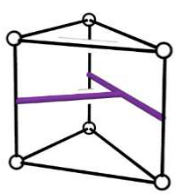


Figure 68 - The periodic cell of Net1

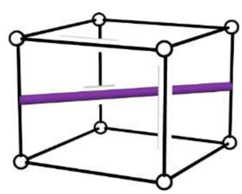
Replicating the E.P.R.s Tet.1, Tet.3, Tet.4, Tet.5, Tet.6 and Tet.7, all from the Tetragonal system, lead to the formation of 2-fold axis networks which are topologically similar to Net1. That is, multi-layered networks with equal distance between layers and rectangular bases (Figure 69).



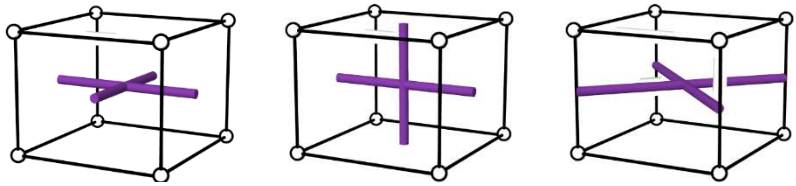
Tet.1



Tet.3



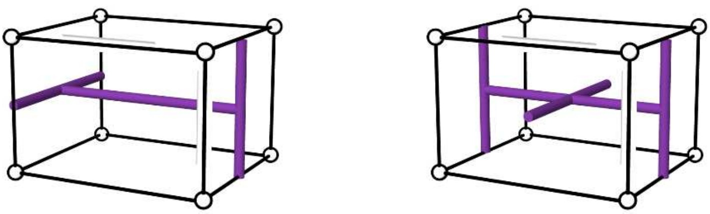
Tet.4



Tet.5 *Tet.6* *Tet.7*

Figure 69 - Additional E.P.R.s which may form Net1

Replicating the E.P.R.s Ortho.4 and Ortho.5 (Figure 70) forms a 2-fold axis network of which axes join at a vertex in a cross-shape. In one direction along the network axes, the crosses in adjacent vertices are perpendicular to one another. In the other two directions, they are parallel to each other (Figure 71).



Ortho.4 *Ortho.5*

Figure 70 - The E.P.R.s Ortho.4 and Ortho.5 along with their 2-fold axes

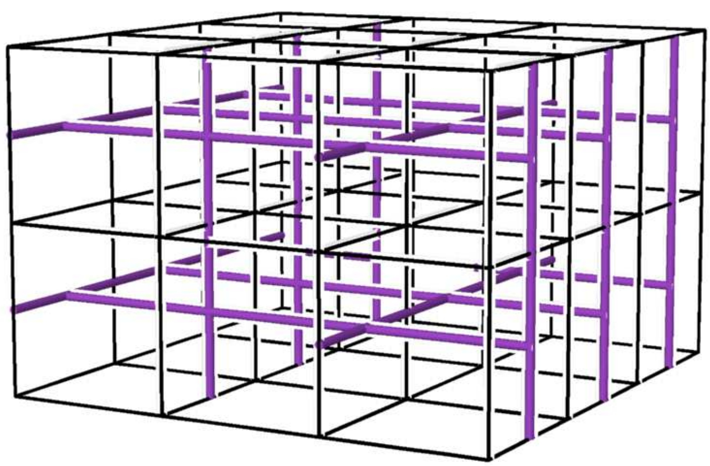


Figure 71 - The 2-fold axis network Net2

The periodic cell of the network, henceforth “Net2”, is a connected polygon with eight edges, in which a saddle-shaped surface segment is bounded (Figure 72).

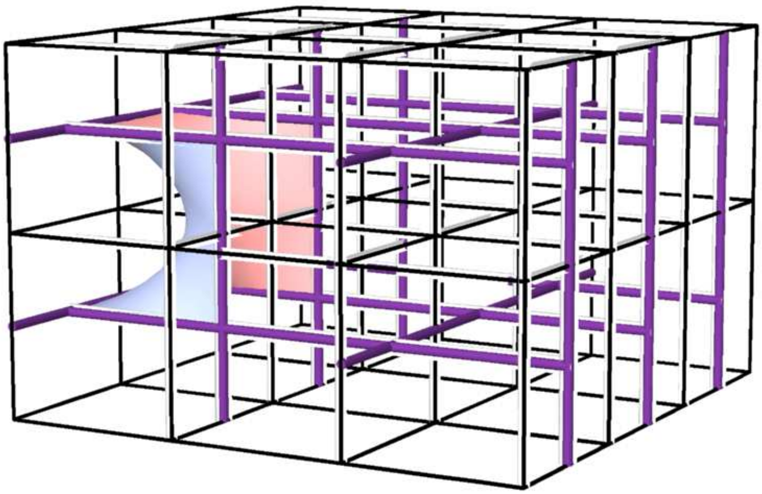


Figure 72 - The periodic cell of Net2

The 2-fold axis network which is formed by replicating the E.P.R. Ortho.6 (Figure 73), henceforth “Net3”, has uniform vertices. The 2-fold axes which intersect at a vertex form a cross. In one direction along the network’s axes, the crosses at adjacent vertices are parallel. In the other two directions, they are perpendicular (Figure 74).

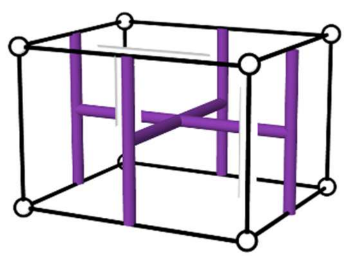


Figure 73 - The E.P.R. Ortho.6 along with its 2-fold axes

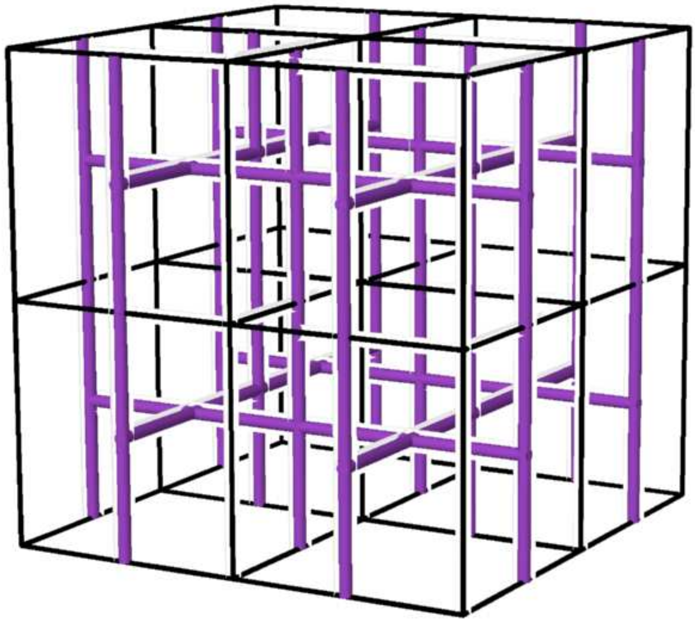
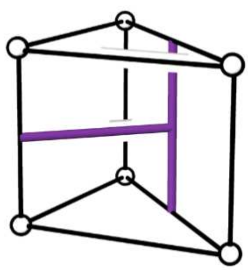
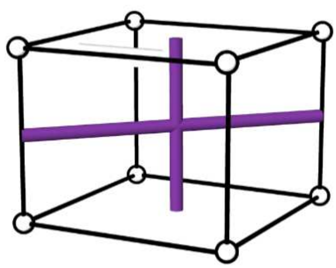


Figure 74 - The 2-fold axis network Net3

The E.P.R.s Tet.2 and Tet.8 (Figure 75) lead to the formation of 2-fold axis networks which are topologically similar to Net3.



Tet.2



Tet.8

Figure 75 - Additional E.P.R.s which may form Net2

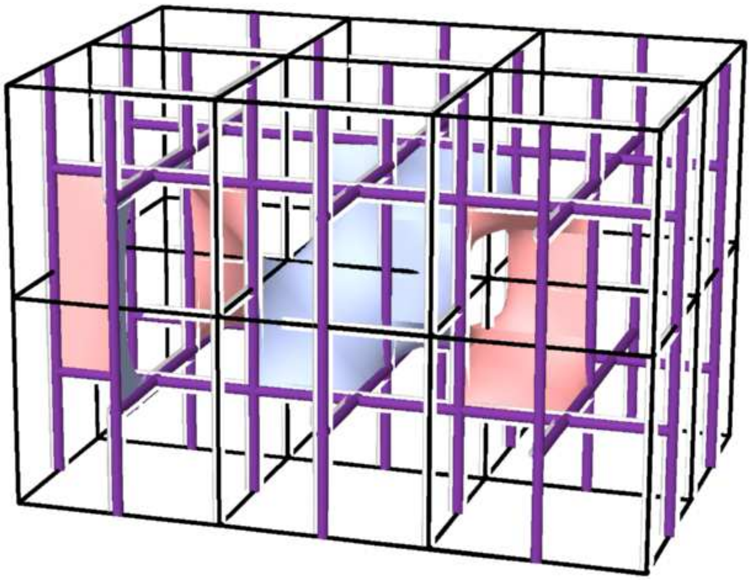


Figure 76 - Three periodic cells within Net3

Within Net3 are three different periodic cells (Figure 76), each of which can lead to the formation of a surface which partitions space into two identical subspaces (Figure 77).

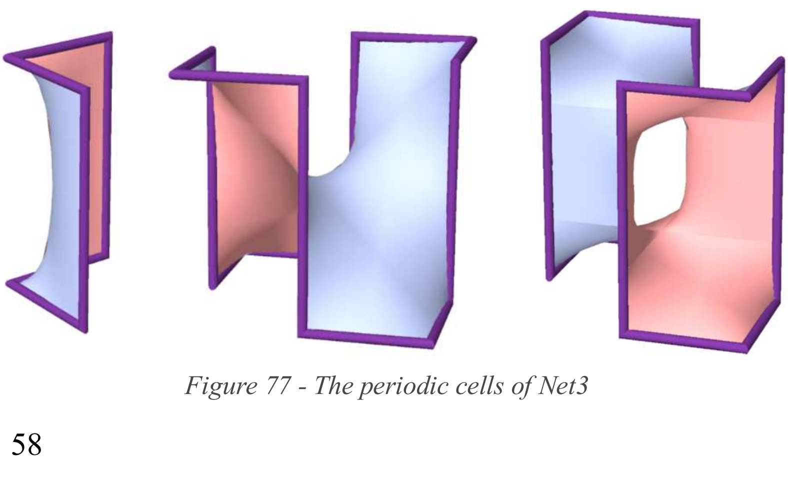


Figure 77 - The periodic cells of Net3

The 2-fold axis network which is formed by the replication of the E.P.R. Tet.9 (Figure 78) is a multi-layered network. Each layer of the network consists of a planar grid of right-angled isosceles triangles (Figure 79).

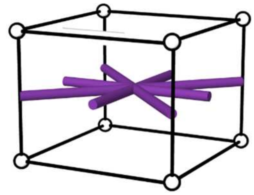


Figure 78 - The E.P.R. Tet.9 along with its 2-fold axes

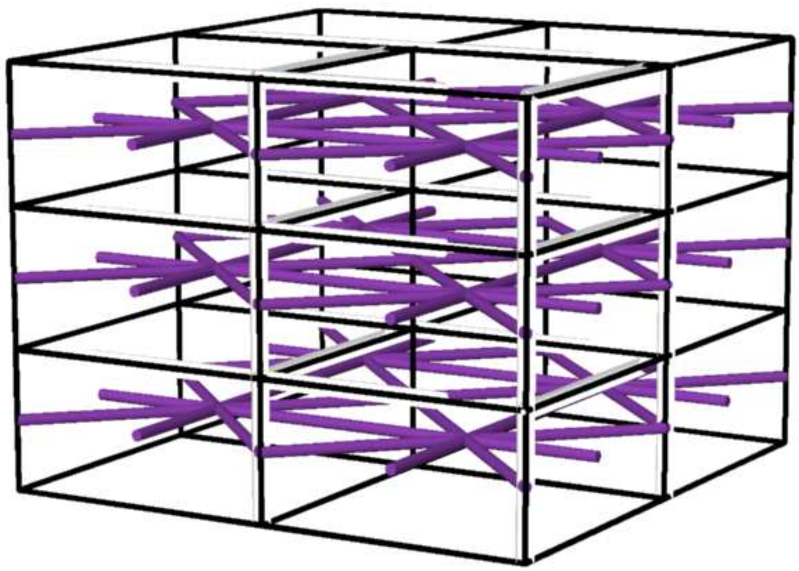


Figure 79 - The 2-fold axis network Net4

The periodic cell of this network, henceforth "Net4", is a split polygon cell with right-angled isosceles triangles as its bases (Figure 80).

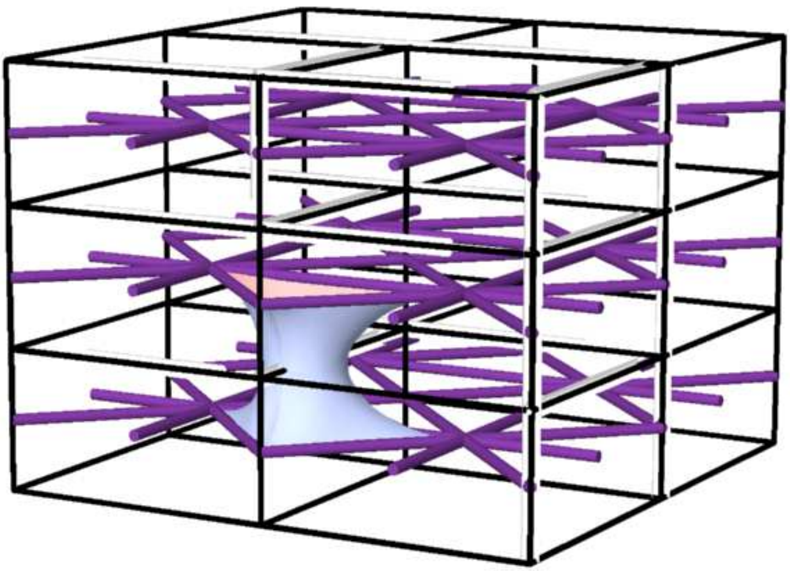


Figure 80 – The periodic cell of Net4

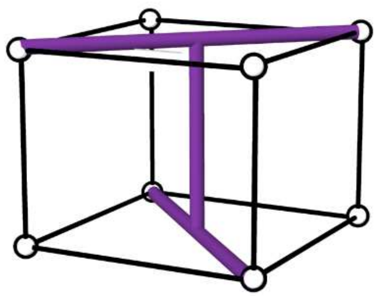


Figure 81 - The E.P.R. Tet.10 along with its 2-fold axes

The 2-fold axis network which is formed by replicating the E.P.R. Tet.10 (Figure 81) has uniform vertices. The 2-fold axes which intersect at a vertex form a cross. Along the network's axes, the crosses along adjacent vertices are perpendicular to each other (Figure 82). The network, henceforth "Net 5", is sometimes referred to as "The Crosses Network".

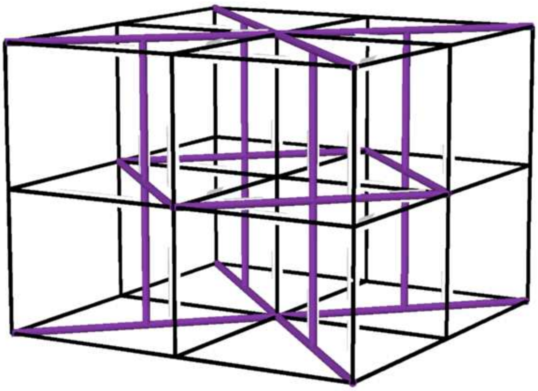


Figure 82 - The 2-fold axis network Net5

Replicating the E.P.R. Cub.5 (Figure 83) which belongs to the Cubic system, leads to the formation of a 2-fold axis network which is topologically similar to Net5 (Figure 84).

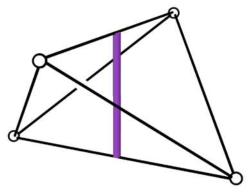


Figure 83 - Additional E.P.R. which may form Net5

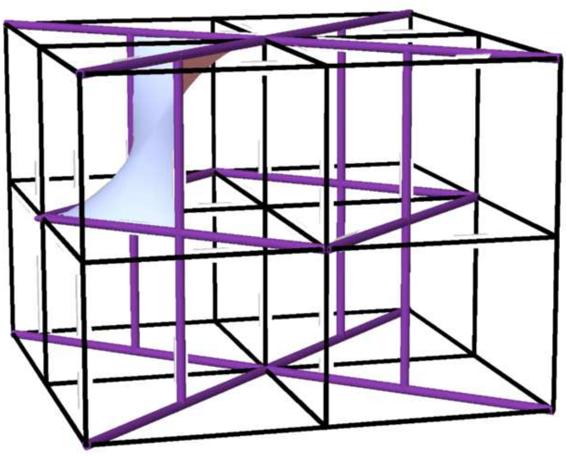


Figure 84 - The periodic cell of Net5

The 2-fold axis network formed by replicating the E.P.R. Tet.11 (Figure 85) has uniform vertices. The 2-fold axes which intersect at a vertex form a cross. The network forms interlaced parallel planar layers. In one layer, the crosses are parallel to each other. In the next, the crosses are perpendicular to each other. Along the axes going perpendicular to the layers, the crosses form a corkscrew shape, with each intersection being rotated 45^0 in relation to the adjacent intersections (Figure 86).

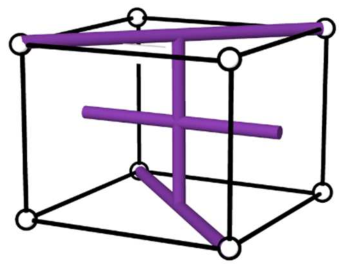


Figure 85 - The E.P.R. Tet.11 along with its 2-fold axes

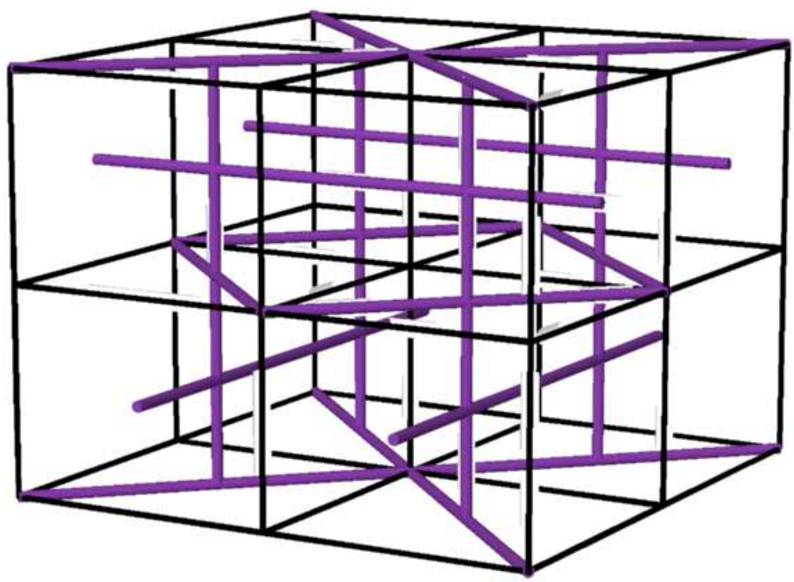


Figure 86 - The 2-fold axis network Net6

This 2-fold axis network, henceforth “Net6”, contains two different periodic cells (Figure 87), each of which may lead to the formation of a surface which partitions space into two identical subspaces (Figure 88).

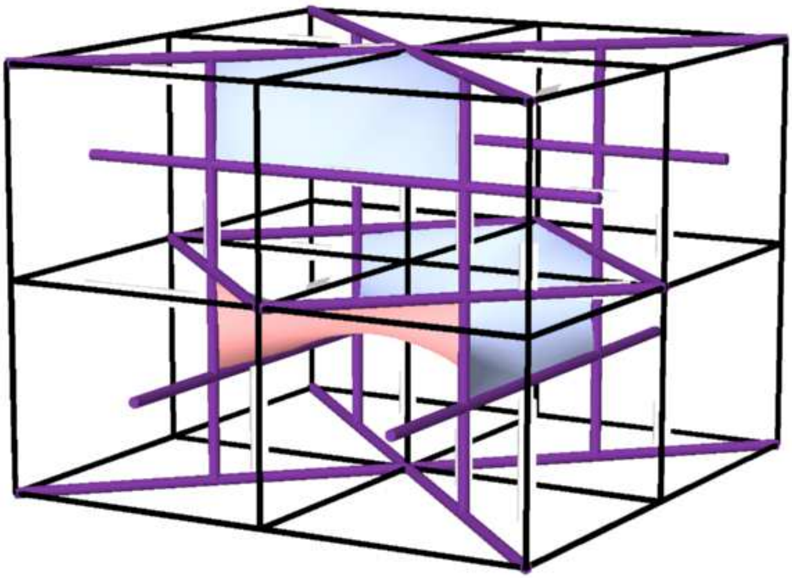


Figure 87 - Two periodic cells within Net6

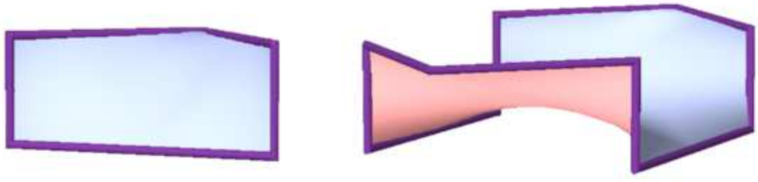


Figure 88 - The periodic cells of Net6

The 2-fold axis network formed by the replication of the E.P.R. Cub.1 (Figure 89) can be described as a combination of three multi-layer networks which are similar to Net1. The directions of the layers of each of these three networks is perpendicular to the other two (Figure 90).

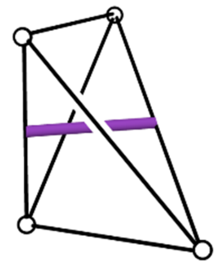


Figure 89 - The E.P.R. Cub.1 along with its 2-fold axis

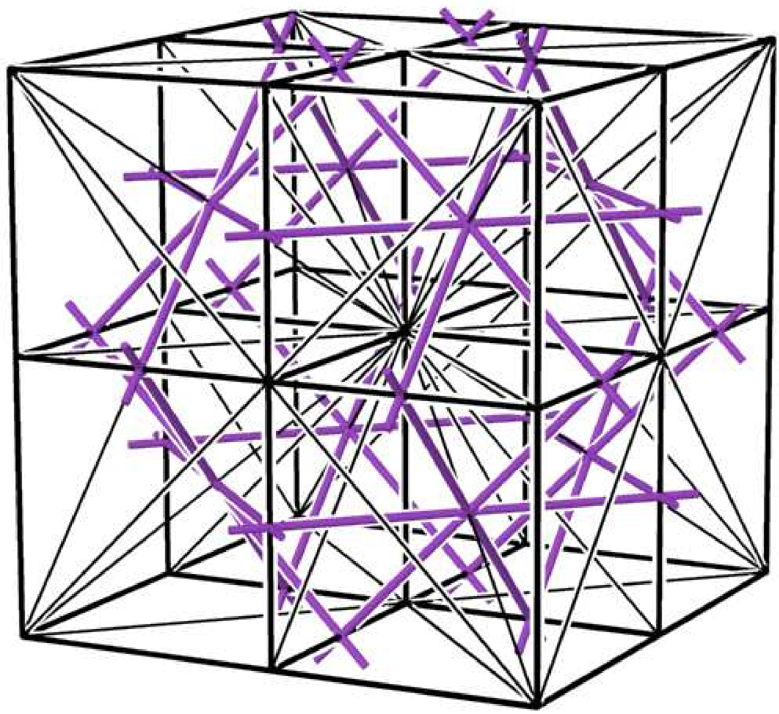


Figure 90 - The 2-fold axis network Net7

Replicating the E.P.R. Cub.2 (Figure 91) leads to a network which is topologically similar to the above network, henceforth "Net7".

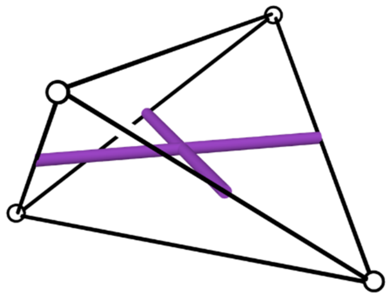


Figure 91 - Additional E.P.R. which may form Net7

Within Net7 there exist two different periodic cells (Figure 92), each of which may lead to the formation of a surface which partitions space into two identical subspaces (Figure 93).

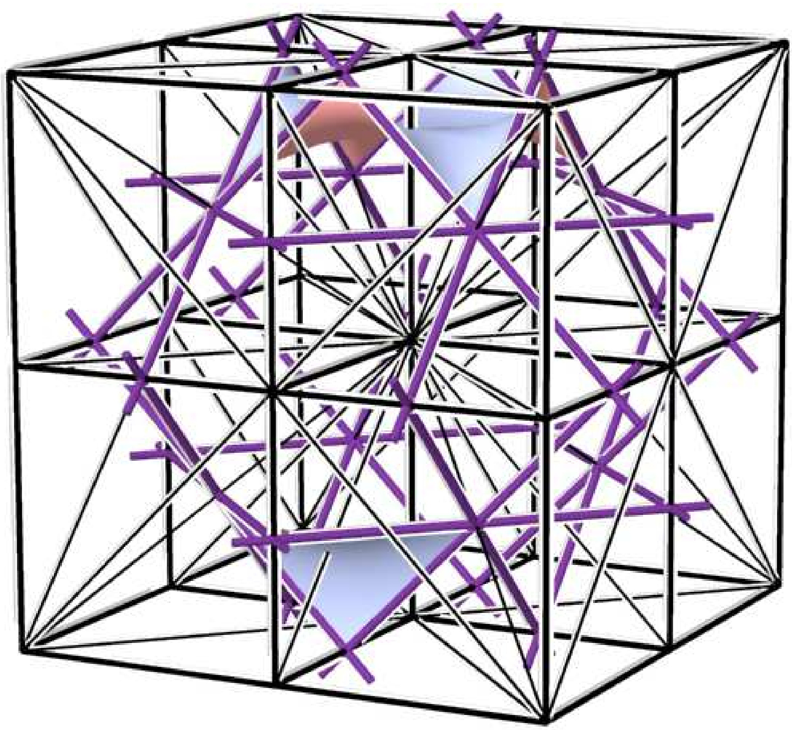


Figure 92 - Two periodic cells within Net7

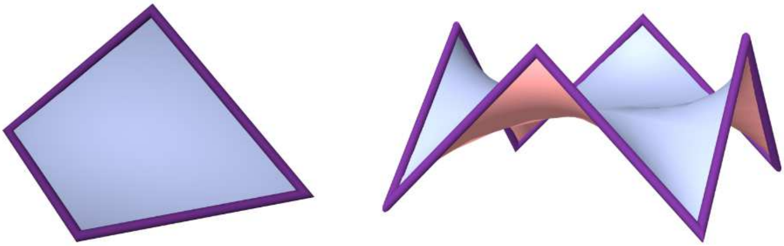


Figure 93 - The periodic cells of Net7

The 2-fold axis network that is formed by replicating the E.P.R. Cub.3 (Figure 94) has two types of vertices. One is across shaped. The other is in the intersection of three 2-fold axes on the same plane. The angle between each pair of adjacent 2-fold axes is 60^0 (Figure 95).

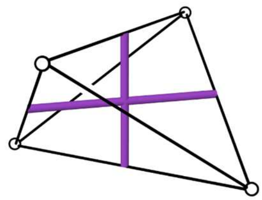


Figure 94 - The E.P.R. Cub.3 along with its 2-fold axes

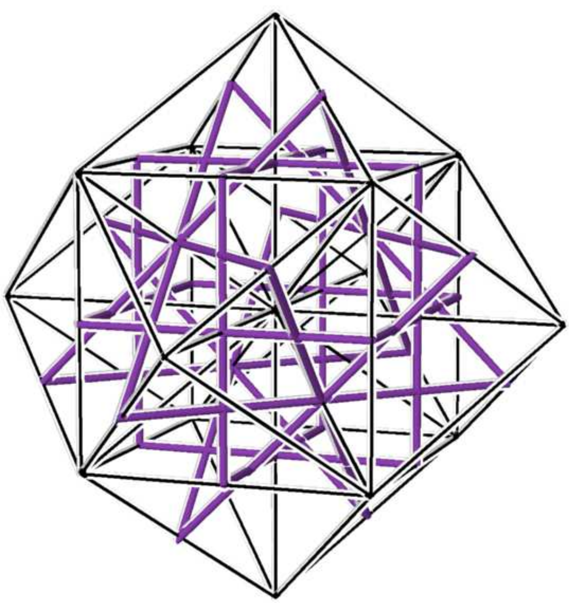


Figure 95 - The 2-fold axis network Net8

The periodic cell of the network, henceforth "Net8", is in the form of a connect polygon with four edges, in which a saddle-shaped surface segment is bounded (Figure 96).

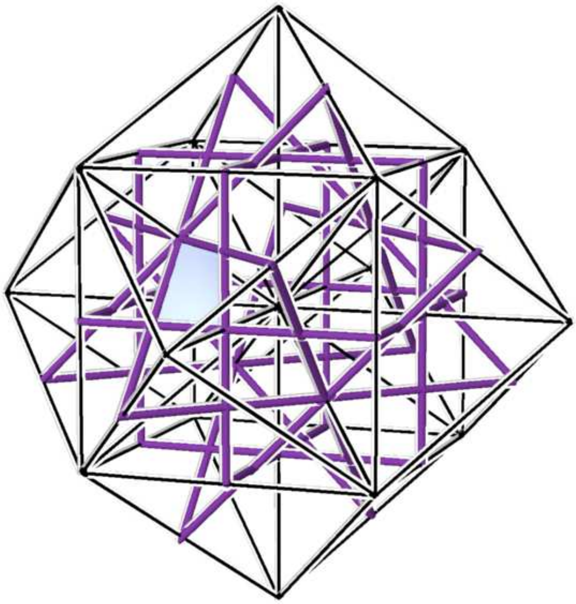


Figure 96 - The periodic cell of Net8

The 2-fold axis network formed by replicating the E.P.R. Cub.4 (Figure 97) has uniform vertices. Each vertex is an intersection of three 2-fold axes. The angle between each pair of adjacent 2-fold axes is 60^0 (Figure 98).

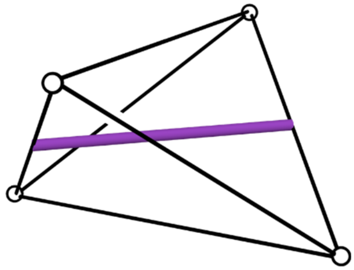


Figure 97 - The E.P.R. Cub.4 along with its 2-fold axis

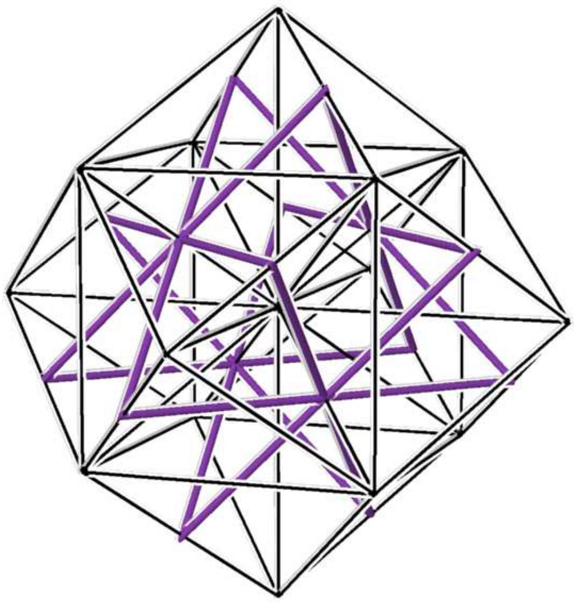


Figure 98 - The 2-fold axis network Net9

This 2-fold axis network, henceforth “Net9”, contains two different periodic cells (Figure 99), each of which may lead to the formation of a surface which partitions space into two identical subspaces.

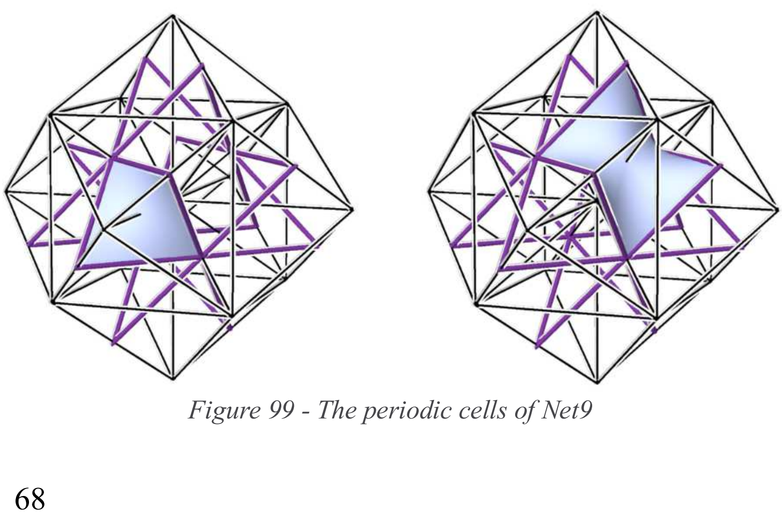


Figure 99 - The periodic cells of Net9

The 2-fold axis network which is formed by the replication of the E.P.R. Tri.1 (Figure 100) is a multi-layered network. Each layer of the network consists of a planar grid of equilateral triangles. Any two adjacent layers are rotated 180^0 relative to one another (Figure 101).

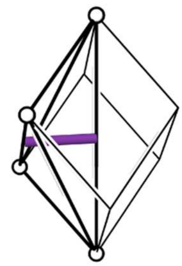


Figure 100 - The E.P.R. Tri.1 along with its 2-fold axis

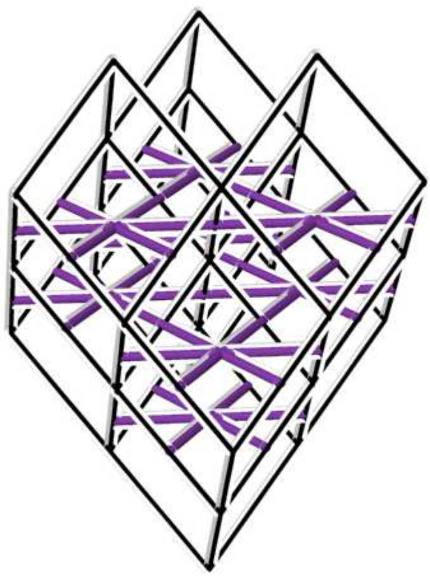


Figure 101 - The 2-fold axis network Net10

The periodic cell of this network, henceforth “Net10”, is a split polygon cell with equilateral triangles, which are rotated 180^0 relative to one another, as its bases (Figure 102).

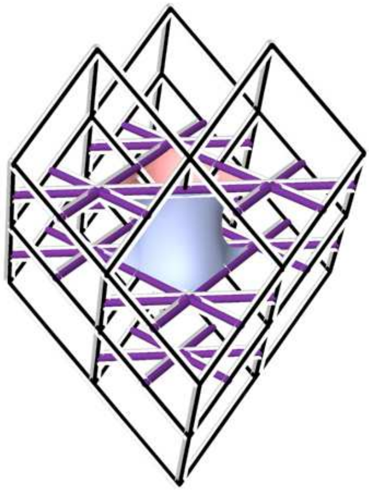


Figure 102 - The periodic cell of Net10

The 2-fold axis network which is formed by the replication of the E.P.R. Hex.1 (Figure 103) is a multi-layered network. Each layer of this network, henceforth "Net11", consists of a planar grid of equilateral triangles (Figure 104).

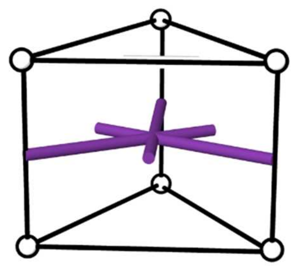


Figure 103 - The E.P.R. Hex.1 along with its 2-fold axes

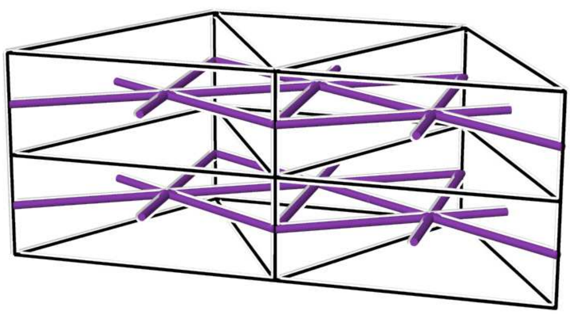
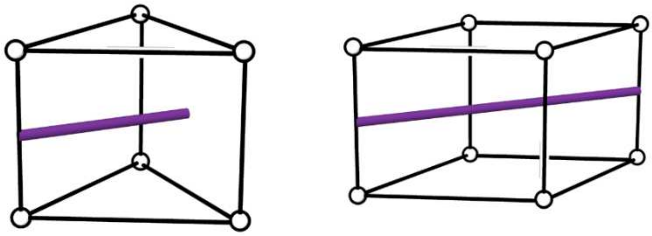


Figure 104 - The 2-fold axis network Net11

Replicating the E.P.R.s Hex.2 and Hex.9 (Figure 105) leads to the formation of 2-fold axis networks which are topologically similar to Net11.



Hex.2 *Hex.9*

Figure 105 - Additional E.P.R.s which may form Net11

The periodic cell of Net11 is a split polygon cell with equilateral triangles as its bases (Figure 106).

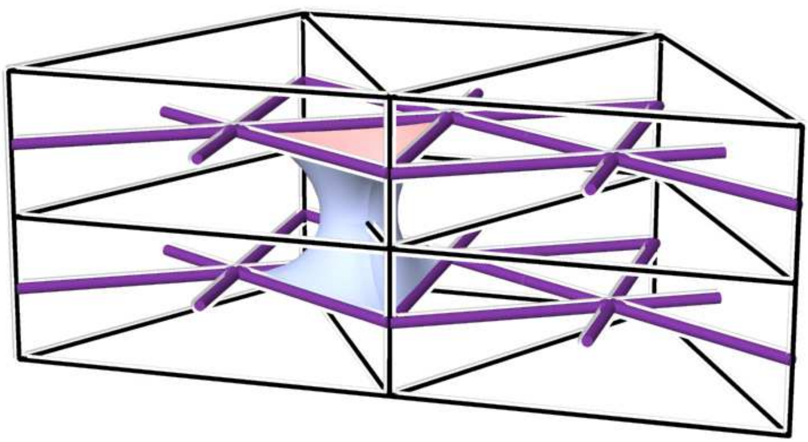


Figure 106 - The periodic cell of Net11

The 2-fold axis network which is formed by the replication of the E.P.R. Hex.3 (Figure 107) is a multi-layered network. Each layer of this network, henceforth “Net12”, consists of a planar grid of triangles with angles which are 30^0, 60^0 and 90^0 (Figure 108).

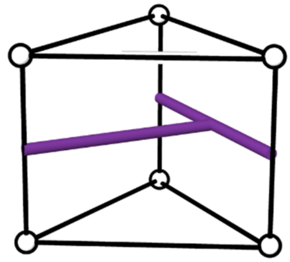


Figure 107 - The E.P.R. Hex.3 along with its 2-fold axes

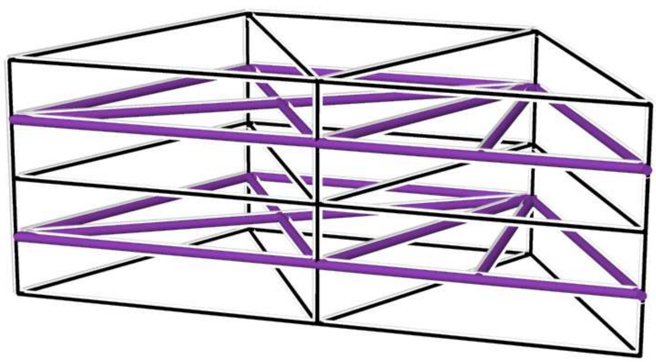


Figure 108 - The 2-fold axis network Net12

Replicating the E.P.R.s Hex.5, Hex.6, Hex.8 and Hex.11 (Figure 109) leads to the formation of 2-fold axis networks which are topologically similar to Net12.

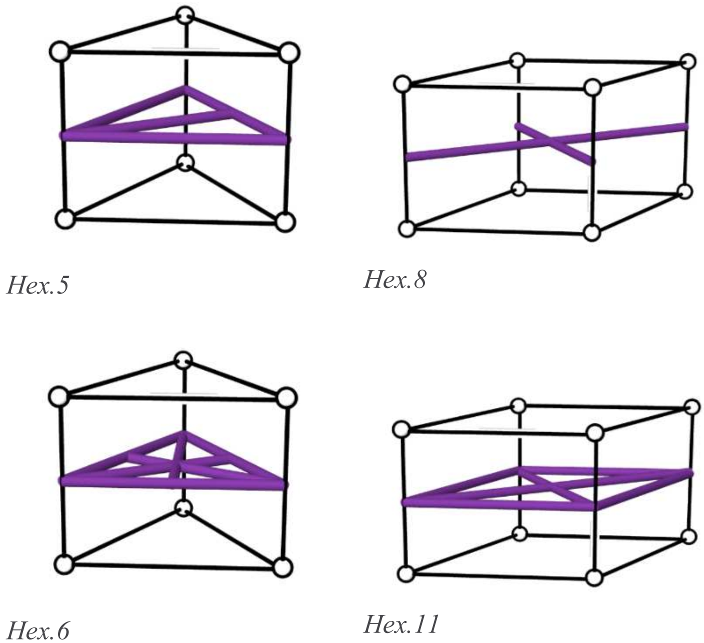


Figure 109 - Additional E.P.R.s which may form Net12

The periodic cell of Net12 is a split polygon cell with triangles with angles which are 30^0, 60^0 and 90^0 as its bases (Figure 110).

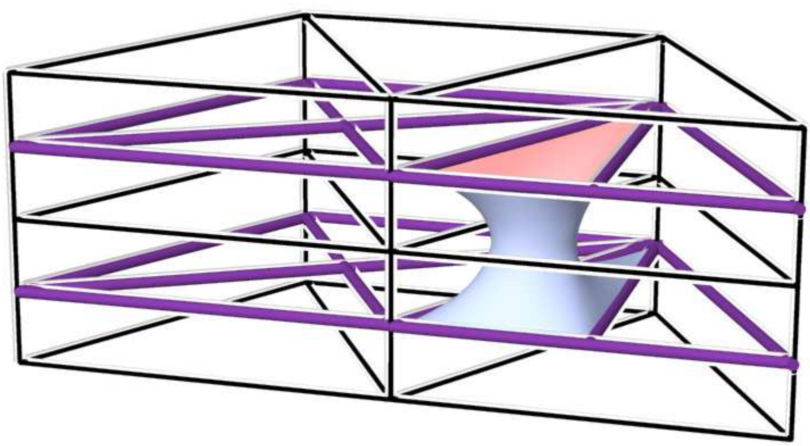


Figure 110 - The periodic cell of Net12

The 2-fold axis network which is formed by the replication of the E.P.R. Hex.4 (Figure 111) is a connected network made of multiple planar layers of equilateral triangles. Perpendicular to the layers are 2-fold axes which go through the middles of the edges of the triangles (Figure 112).

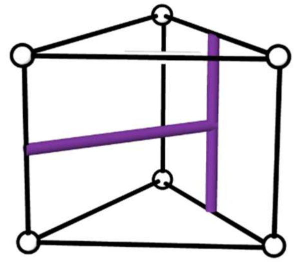


Figure 111 - The E.P.R. Hex.4 along with its 2-fold axes

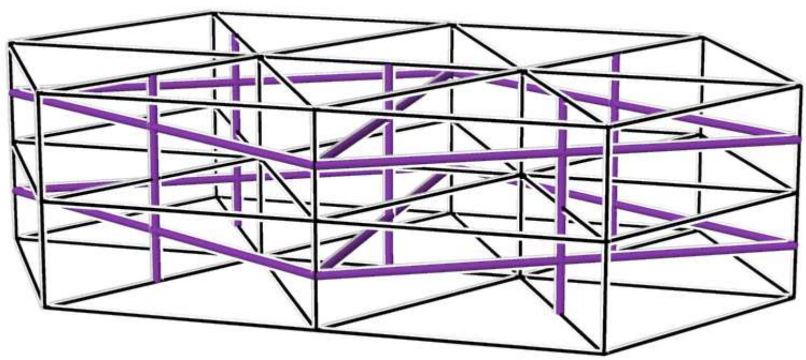


Figure 112 - The 2-fold axis network Net13

Replicating the E.P.R. Hex.10 (Figure 113) leads to the formation of a 2-fold axis network which is topologically similar to the above network, henceforth “Net13”.

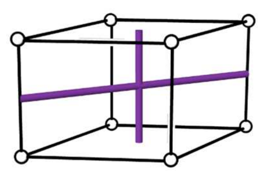


Figure 113 - E.P.R Hex.10 which may form Net13

Any periodic cell within Net13, in which a minimal surface segment may be bounded, would, if replicated, lead to the creation of a surface which self-intersects. Therefore, it does not contain a periodic cell which may lead to the formation of a surface which partitions space into two identical subspaces.

The 2-fold axis network which is formed by the replication of the E.P.R. Hex.7 (Figure 114) is a connected network made of multiple planar layers of equilateral triangles. Perpendicular to the layers are 2-fold axes which go through the middles of the edges of the triangles (Figure 115).

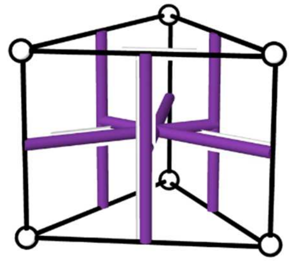


Figure 114 - The E.P.R. Hex.7 along with its 2-fold axes

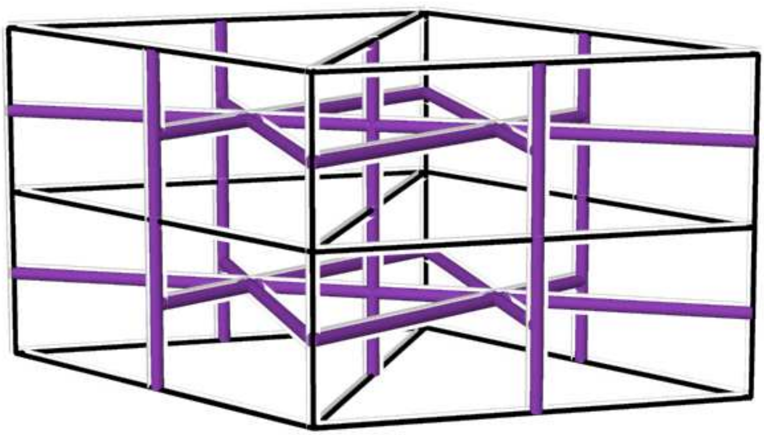


Figure 115 - The 2-fold axis network Net14

This network, henceforth "Net14", also does not contain a periodic cell which may lead to the formation of a surface which partitions space into two identical subspaces.

5.5 Surface construction and topological classification

In the previous chapter, we have discovered fourteen topologically different 2-fold axis networks. In two of the networks, no E.P.R. in which a surface segment which may lead to the construction of a smooth periodic surface which partitions space into two identical subspaces could be found. Within the remaining twelve 2-fold axis networks, seventeen suitable E.P.R.s were found. Replicating the elementary surface segment using the 2-fold axes in which it is bounded leads to the creation of a smooth periodic surface which divides space into two identical subspaces.

However, being constructed from distinct E.P.R.s does not guarantee that the surfaces are topologically distinct. As mentioned in the chapter describing the characteristics of the surfaces, the two subspaces into which a surface partitions space can be described using two dual-complementary tunnel networks. These tunnel networks can be used to topologically classify the surfaces. If two surfaces have topologically identical tunnel networks, then they are topologically identical.

These topologically distinct dual networks will be named based on their typical structure, and using characteristic names, some of which are known.

The E.P.R.s are replicated based on the order in which they were located, until receiving a surface segment which allows identifying the tunnel network. After identifying the tunnel network, we shall point out other E.P.R.s which may form topologically identical networks. The typical surface that is attributed to a typical tunnel network is the surface with the largest degree of symmetry. Meaning, the symmetry group which operates on it has the maximal number of symmetry elements. Additionally, it has the smallest bounding cell.

The region containing the typical surface segment is derived based on the elements of the symmetry group which defines it, until receiving the E.P.R. which represents the region, the 2-fold axis network, the surface, and the tunnel network.

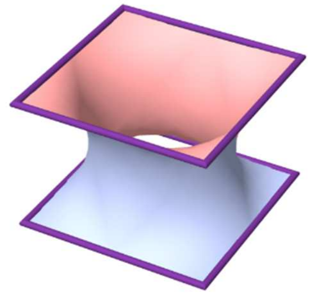


Figure 116 - The split polygon cell of Net1

Replicating the split polygon cell of Net1 (Figure 116) yields a surface (Figure 117), the tunnel network of which is topologically similar to a cubic network (Figure 118). We refer to this surface as the “cubic surface”.

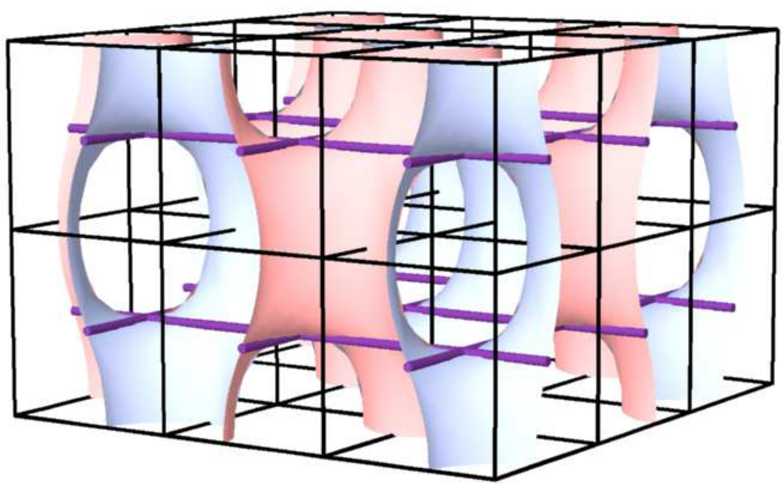


Figure 117 - The cubic surface

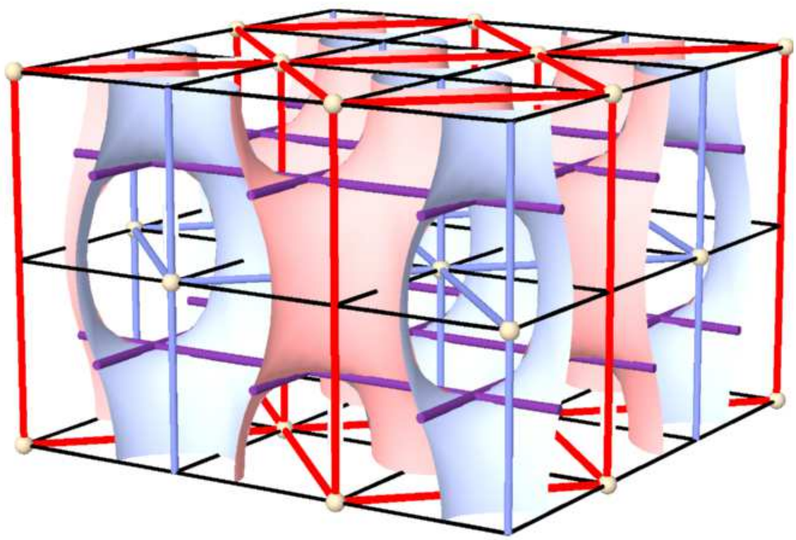


Figure 118 - Tunnel networks of the cubic surface

There are two more periodic cells (Figure 119) which lead to the formation of surfaces which are topologically similar to this surface. One is a cell derived from Net7, the other from Net9. Both of these 2-fold axis networks are defined by the cubic system. The first cell is the smallest cell which forms this surface, and is the elementary periodic unit which is bounded by the 2-fold axes. This typical surface partitions two cubic networks (Figure 120).

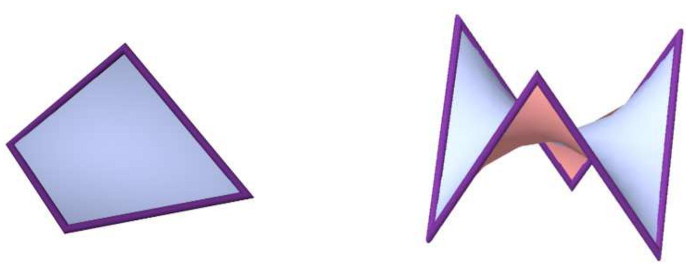


Figure 119 - Additional periodic cells which lead to the formation of the cubic surface

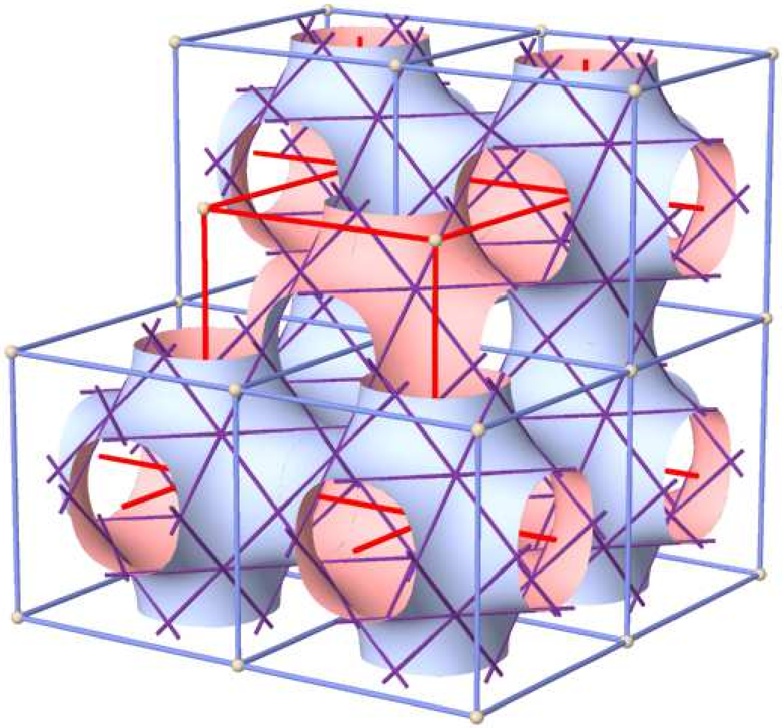


Figure 120 - Cubic networks separated by the cubic surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 121).

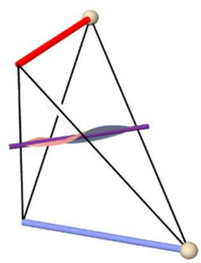


Figure 121 - Minimal E.P.R. representing the cubic surface

Replicating the connected eight-edged polygon periodic cell of Net2 (Figure 122) leads to the formation of a surface (Figure 123), the tunnel network of which is topologically similar to the diamond network (Figure 124). We refer to this surface as the "diamond network surface".

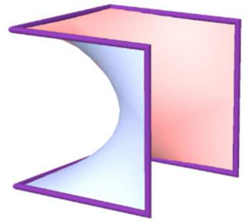


Figure 122 – The connected eight-edged polygon periodic cell of Net2

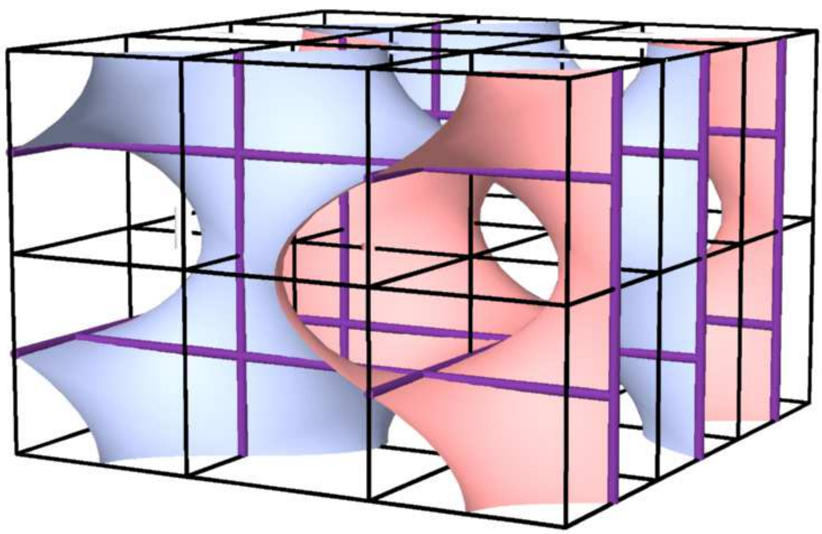


Figure 123 - The diamond network surface

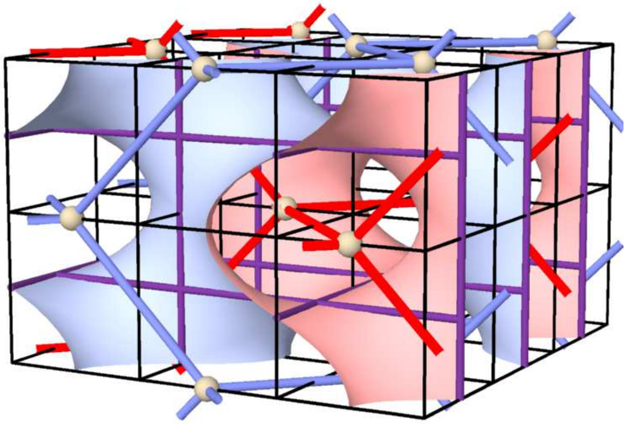


Figure 124 - Tunnel networks of the diamond network surface

There are seven additional periodic cells (Figure 125) which lead to the formation of a surface that is topologically similar. Two are derived from Net6, while the rest are derived from Net3, Net5, Net7, Net9, and Net10. These 2-fold axis networks are defined by the cubic system. The cell derived from the 2-fold axis network Net8 is the smallest cell which forms this surface. It is the elementary periodic unit which is bounded by the 2-fold axes. This typical surface partitions two diamond networks (Figure 126).

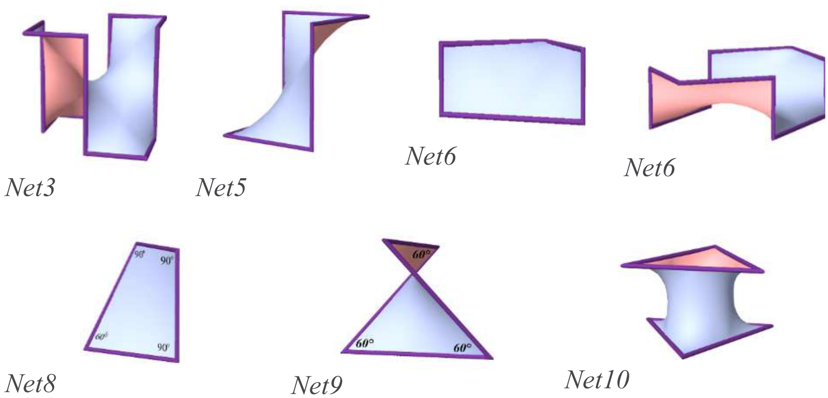


Figure 125 - Additional periodic cells which lead to the formation of the cubic surface

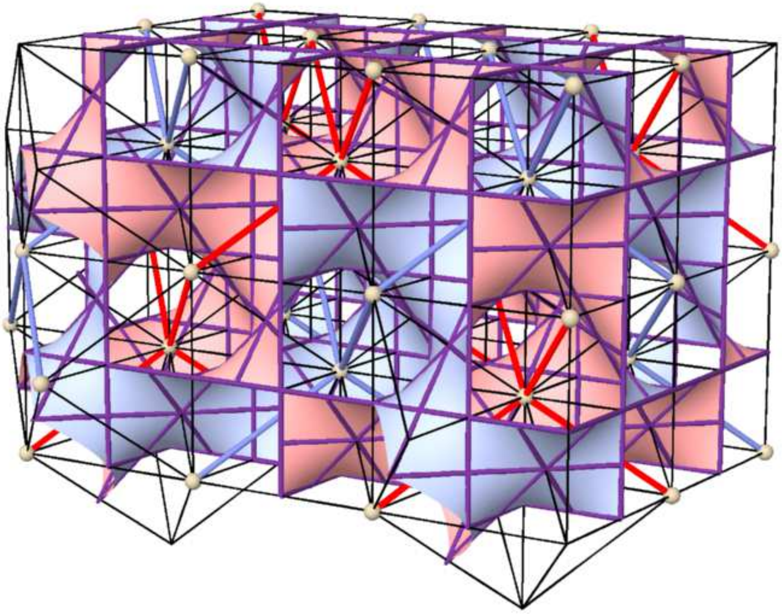


Figure 126 - Diamond networks separated by the diamond network surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 127).

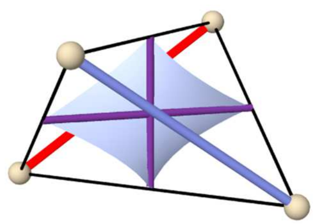


Figure 127 - Minimal E.P.R. representing the diamond network surface

Replicating the connected six-edged polygon periodic cell of Net3 (Figure 128) leads to the formation of a surface (Figure 129), the tunnel network of which has uniform vertices (Figure 130). Each vertex is the intersection of four edges on one plane. The planes of adjacent vertices in one direction are perpendicular to each other. In the other two directions, they are parallel. We refer to this network as the “crosses network” and to the surface which partitions the two networks as the “crosses network surface”.

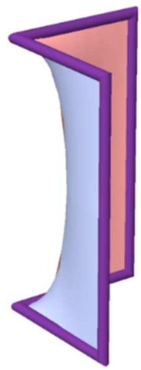


Figure 128 - The connected six-edged polygon periodic cell of Net3

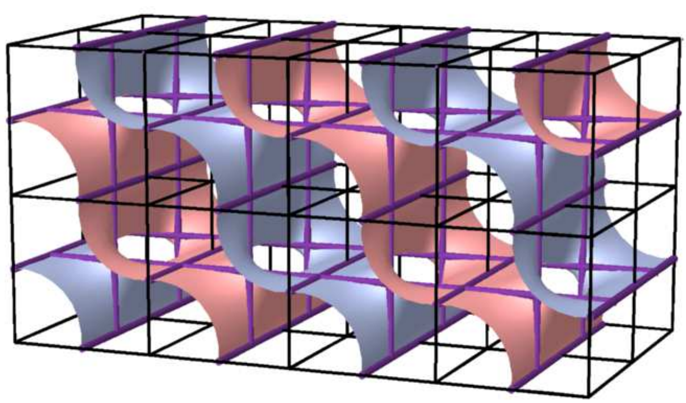


Figure 129 - The crosses network surface

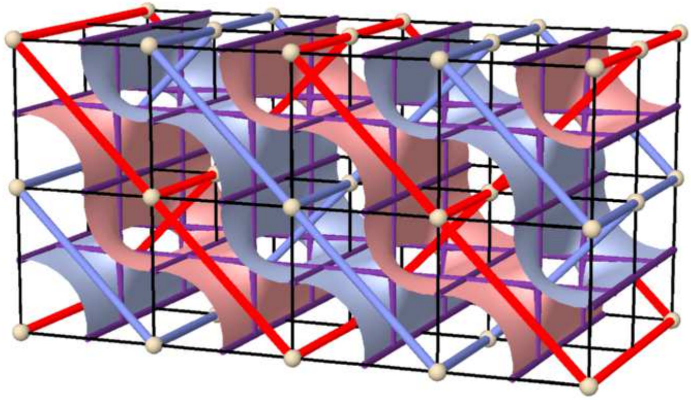


Figure 130 - Network tunnels of the crosses network surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 131).

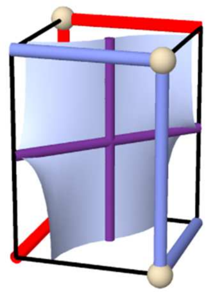


Figure 131 - Minimal E.P.R. representing the crosses network surface

Replicating the split polygon cell of Net3 (Figure 132) yields a surface (Figure 133), the tunnel network of which appears as a combination between a cubic network and a diamond network (Figure 134). The network has two different types of vertices. One is an intersection between four edges, similar to the diamond network, and the other is the intersection of six edges,

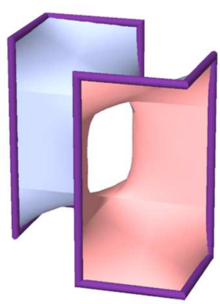


Figure 132 - The split polygon cell of Net3

similar to the cubic network. We refer to this network as the "diamond cubic network" and to the surface which partitions the two networks as the "diamond cubic surface". **The existence of this surface was not known prior to the use of the method described in this article.**

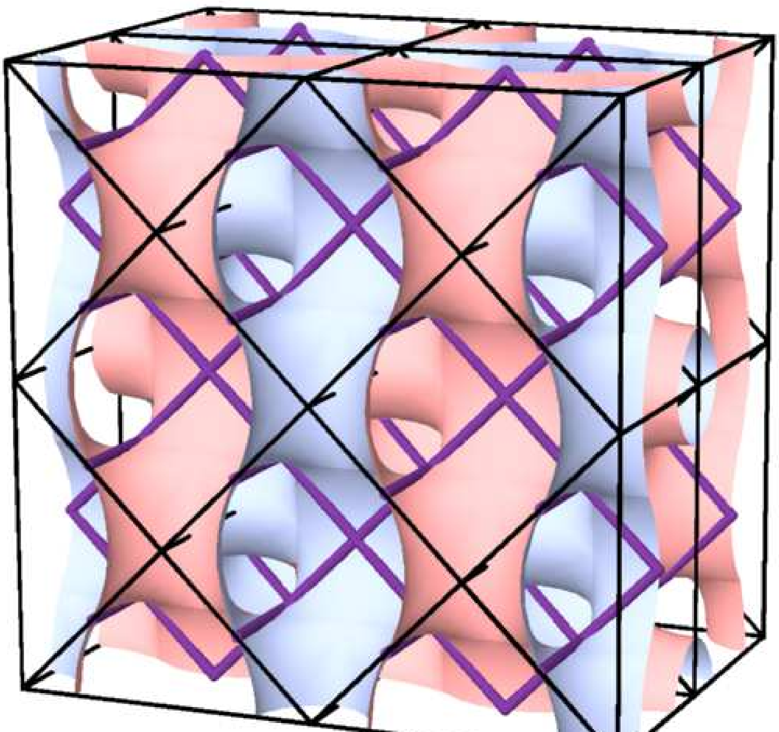


Figure 133 - The diamond cubic surface

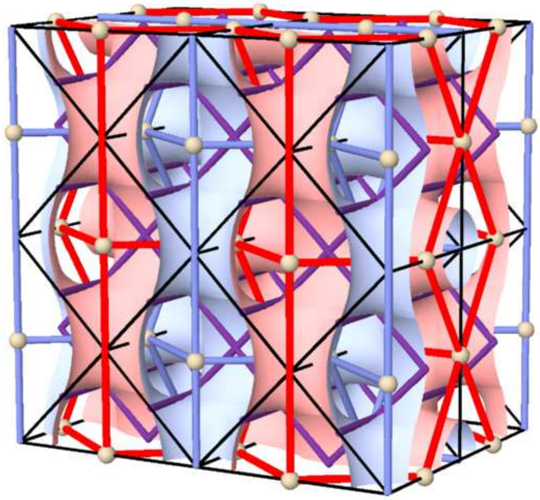


Figure 134 - Tunnel networks of the diamond cubic surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 135).

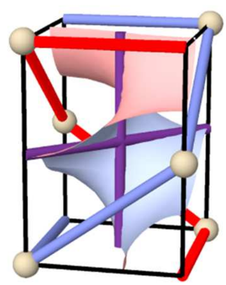


Figure 135 - The minimal E.P.R. representing the diamond cubic surface

Replicating the split polygon cell of Net4 (Figure 136) yields a surface (Figure 137), with a multi-layer tunnel network (Figure 138).Each layer is a grid of pentagons with non-uniform edge lengths. Perpendicular axes connect the different layers. The network has two types of vertices. One is the intersection of four planar

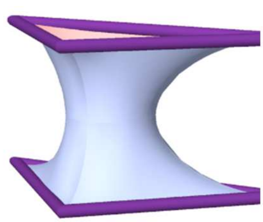


Figure 136 - The split polygon cell of Net4

edges. The other is the intersection of five edges, three of which are in the plane of the pentagons, and the other two are perpendicular to it.

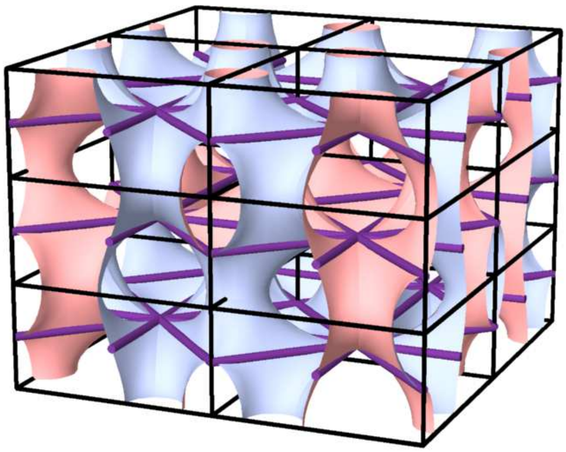


Figure 137 - The 45^0, 90^0, 45^0 triangle surface

Since this surface is more easily described through its 2-fold axis network than its tunnel network, we refer to this surface as the "45^0, 90^0, 45^0 triangle surface".

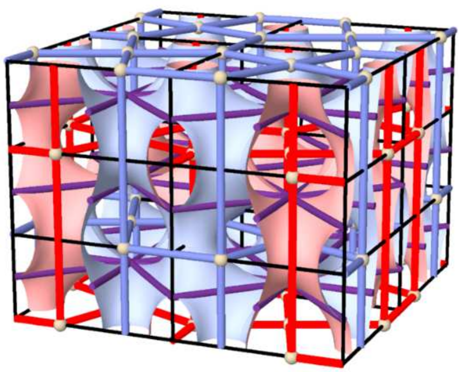


Figure 138 - Tunnel networks of the 45^0, 90^0, 45^0 triangle surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 139).

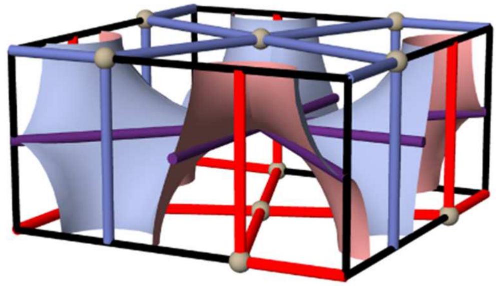


Figure 139 - Minimal E.P.R. representing the 45^0, 90^0, 45^0

Replicating the connected eight-edged non-planar polygon periodic cell of Net7 (Figure 140) leads to the formation of a surface (Figure 141), the tunnel network of which has its axes go through the diagonals of the faces of a tightly packed array of cubes (Figure 142). The network has two types of vertices. One type is at the intersection of four planar edges, located at the center of the faces of the cubes. The other is at the intersection of twelve edges, located at a common vertex to eight cubes, and connecting with the centers of their faces. We refer to this surface as the "face centered surface".

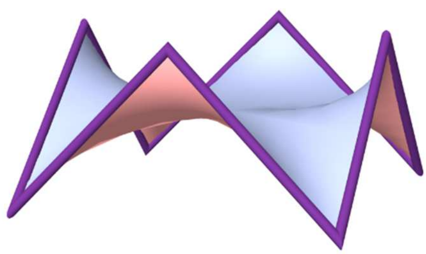


Figure 140 - The connected eight-edged non-planar polygon periodic cell of Net7

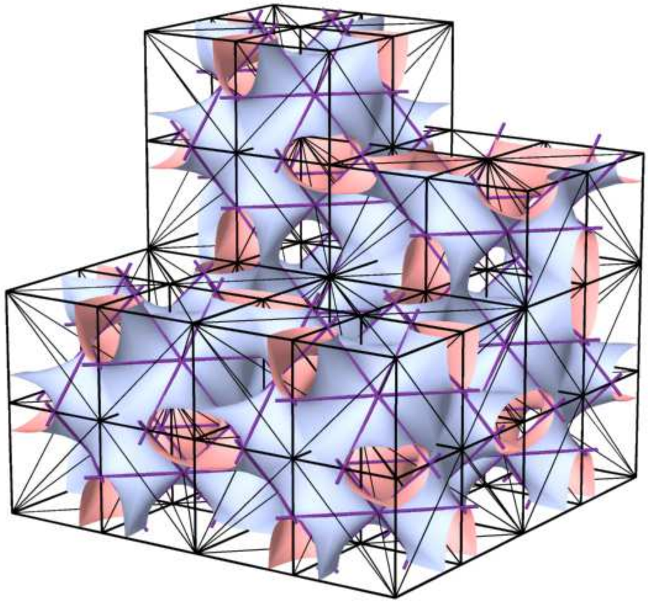


Figure 141 - The face centered surface

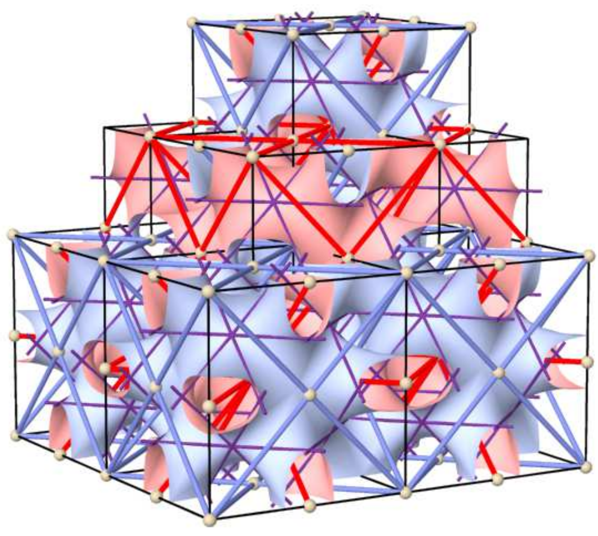


Figure 142 - Tunnel networks of the face centered surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 143).

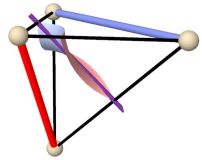


Figure 143 - Minimal E.P.R. representing the face centered

Replicating the split polygon cell of Net11 (Figure 144) yields a surface (Figure 145). with a multi-layer tunnel network (Figure 146).

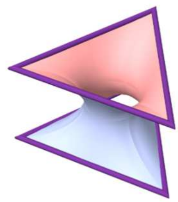


Figure 144 - The split polygon cell of Net11

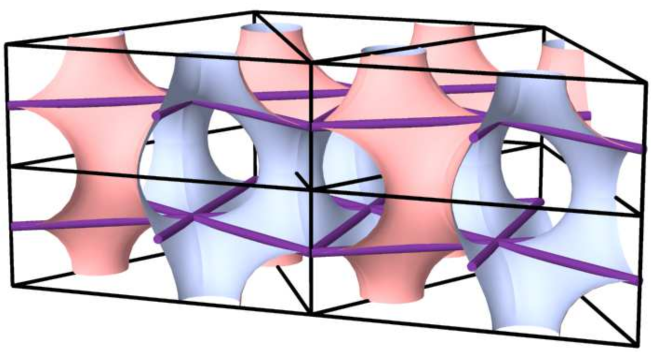


Figure 145 - The pentahedral-trihedral surface

Each layer is a grid of hexagons with uniform edge lengths. Perpendicular axes connect the different layers. The network has two types of vertices. One is the intersection of three planar edges at 120^0 angles from each other. The other is the intersection of five edges, three of which are in the plane of the hexagons, and the other two are perpendicular to it. This surface is referred to as the "pentahedral-trihedral surface".

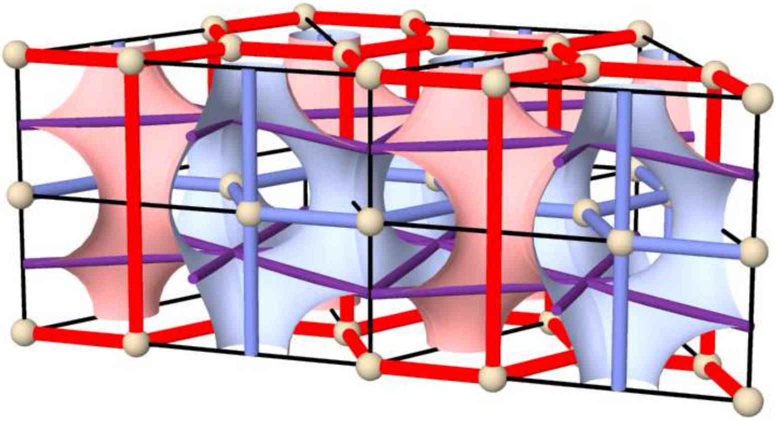


Figure 146 - Tunnel networks of the pentahedral-trihedral surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 147).

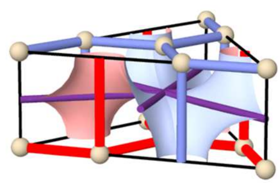


Figure 147 - Minimal E.P.R. representing the pentahedral-trihedral surface

Replicating the split polygon cell of Net12 (Figure 148) yields a surface (Figure 149), with a multi-layer tunnel network (Figure 150). Each layer is a grid of pentagons with non-uniform edge lengths. Perpendicular axes connect the different layers.

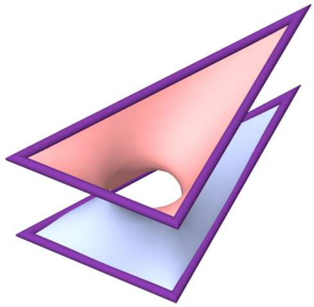


Figure 148 - The split polygon cell of Net12

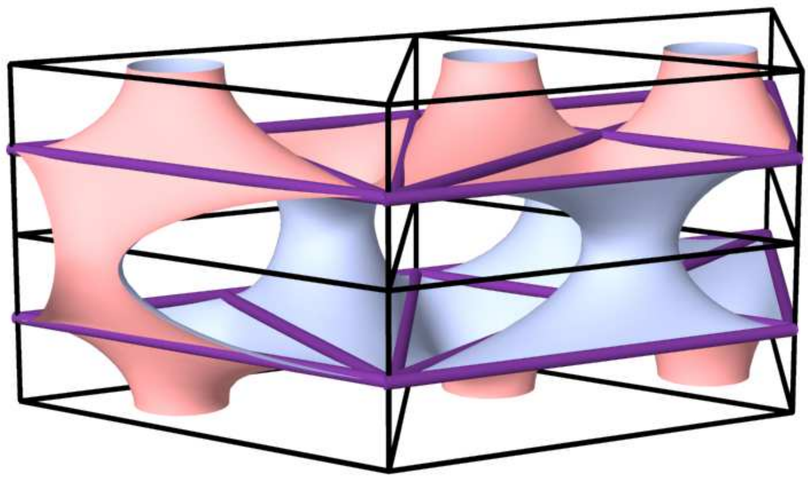


Figure 149 - The 30^0, 60^0, 90^0 triangle surface

The network has three types of vertices. One is the intersection of three planar edges. Another is at the intersection of six planar edges. The last is the intersection of five edges, three of which are in the plane of the pentagons, and the other two are perpendicular to it. Since this surface is more easily described through its 2-fold axis network than its tunnel network, we refer to this surface as the "30^0, 60^0, 90^0 triangle surface".

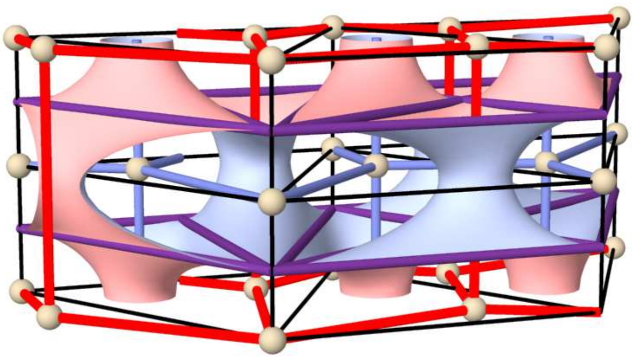


Figure 150 - Tunnel networks of the 30^0, 60^0, 90^0 triangle surface

We divide the space containing the surface by the symmetry elements until receiving the minimal E.P.R. which represents the space, surface, 2-fold axis network and dual-complementary tunnel networks (Figure 151).

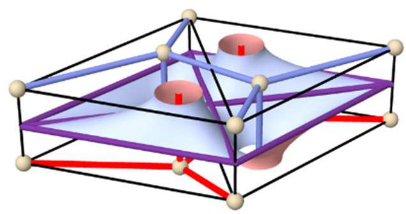
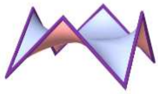


Figure 151 - The minimal E.P.R. representing the 30^0, 60^0, 90^0 triangle surface

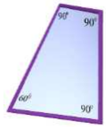
In this process of forming surfaces by replicating periodic cells with minimal surface segments, and topologically categorizing them, we have found eight topologically distinct surfaces. That is, their tunnel networks are topologically distinct.

Each of the eight surfaces can be represented by the smallest periodic cell (Figure 152).

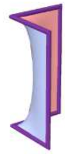
The cubic surface



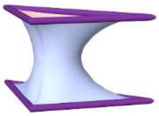
The face centered surface



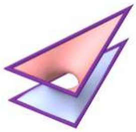
The diamond network surface



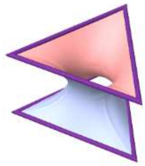
The crosses network surface



The 45^0,90^0, 45^0 triangle surface



The 30^0, 60^0, 90^0 triangle surface



The pentahedral-trihedral surface

The diamond cubic surface

Figure 152 - Periodic cells representing the eight topologically different surfaces

5.6 A new class of surfaces which partition space into two identical subspaces

In the chapter dealing with the properties of the surfaces, we have mentioned that any motif that is bounded by the periodic cells of the 2-fold axis network can be replicated into a surface which partitions space into two identical subspaces. In the previous chapter, we have located eight such surfaces, by bounding a "soap solution surface" in the periodic cells. That is, we located a surface that is formed by dipping the perimeter of the periodic cell in a soap solution. In the next phase, we will consider whether other smooth surfaces can be bounded by the same perimeter, and whether such surfaces could lead to topologically different surfaces which partition space into two identical subspaces. That is, whether they produce tunnel networks which are topologically distinct from the eight already discovered.

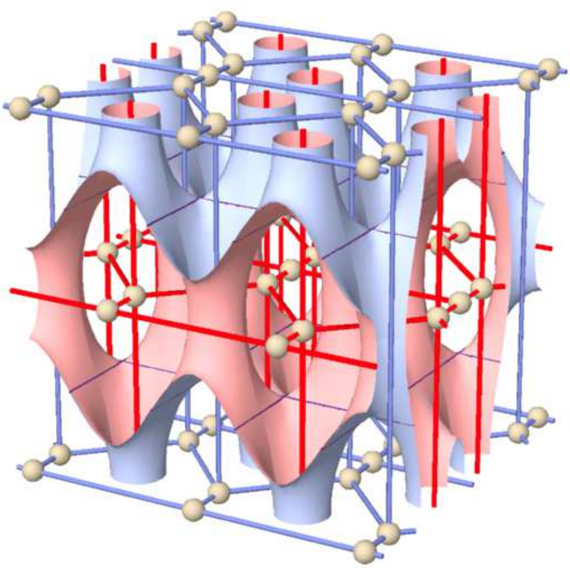


Figure 153 - A surface which partitions space into two identical subspaces, and has a tunnel network that is topologically distinct from the eight already discovered

The question of the possibility for such surface segments to exist arises from the discovery of a surface which partitions space into two identical subspaces, and has a tunnel network that is topologically distinct from the eight already discovered (Figure 153).

This surface was located empirically, by searching for dual-complementary networks. The periodic cell of the surface is a split polygon cell (Figure 154), and is identical to one of the E.P.R.s previously located with one main difference: The surface segment within it is smooth but cannot be formed by dipping the perimeter in a soap solution.

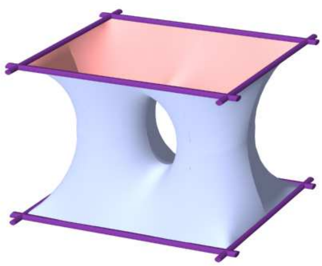


Figure 154 - A periodic cell bounded in a different manner within a known perimeter

Smooth surface segments of this type can only be bounded in split polygon cells. Out of the seventeen periodic cells found, six are split polygon (Figure 155).

The basic shape of a soap solution surface segment bounded by a split polygon cell, is that of a hollow cylinder which connects the two polygons. The new surface segment is in the shape of two such cylinders. In the same manner, surfaces in the shape of three cylinders, four, or more, may be bounded in the same perimeter.

Each of the resulting cells leads to the formation of a surface which divides space into two identical subspaces, which has a topologically unique tunnel network.

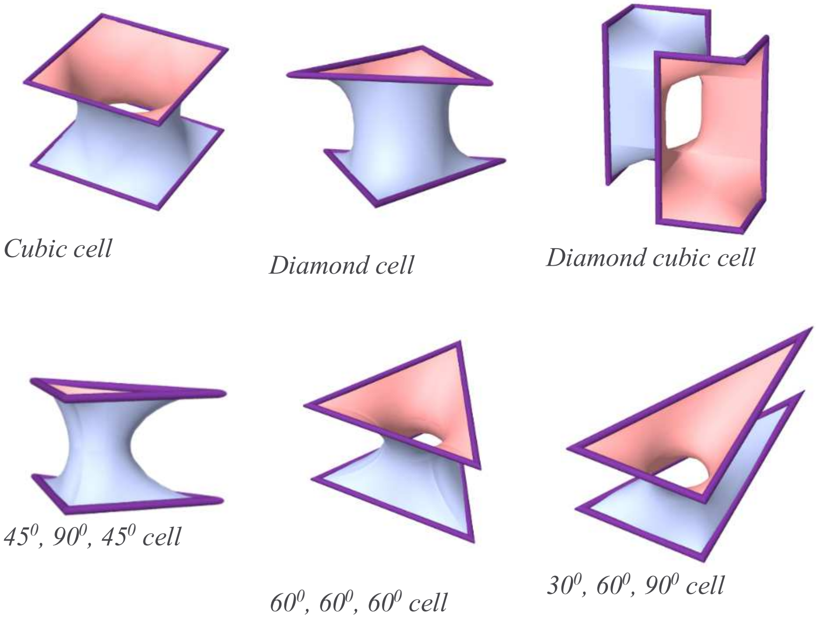


Figure 155 - Split-polygon periodic cells

It may appear that dipping the perimeter in a soap solution cannot result in forming these new surface segments. However, farther dividing them by the mirror symmetry within them can result in a segment which can be formed by a soap solution (Figure 156). This surface segment is bounded by a connected three-dimensional polygon, of which some of the edges are 2-fold axes, and others are the intersection of the surface with the mirror planes of the symmetry group.

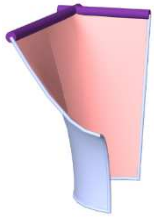


Figure 156 - A surface segment which can be formed by a soap solution

This new class of surfaces includes an infinite number of surfaces, and is unique and distinct from the eight surfaces previously located. We call this class the “multi-sleeved class”. We can demonstrate multiple examples of surfaces which belong to this class (Figures 157 through 160).

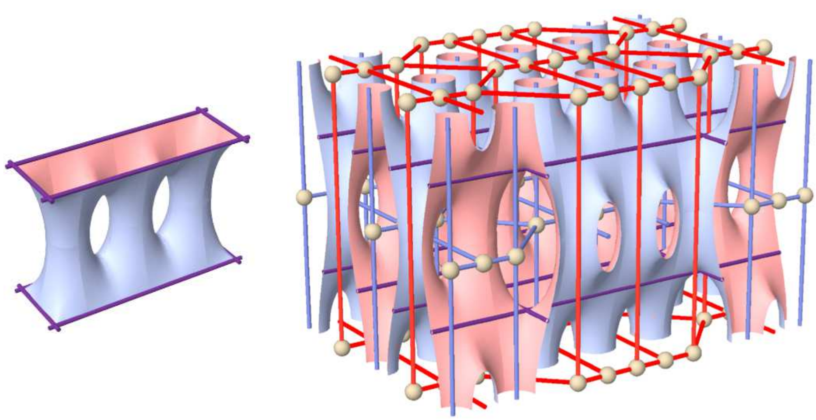


Figure 157 - A cell with orthorhombic symmetry and three sleeves and the surface formed by it

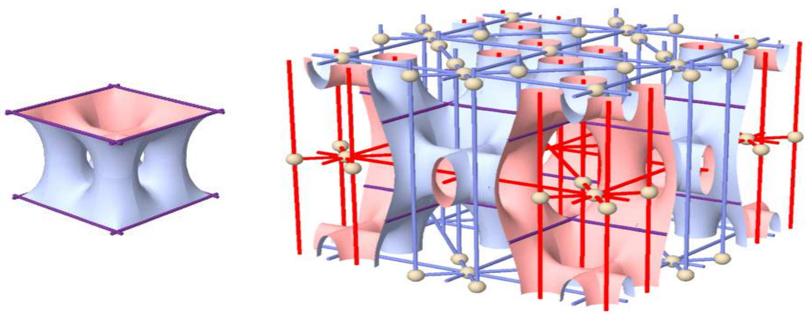


Figure 158 - A cell with tetragonal symmetry and four sleeves and the surface formed by it

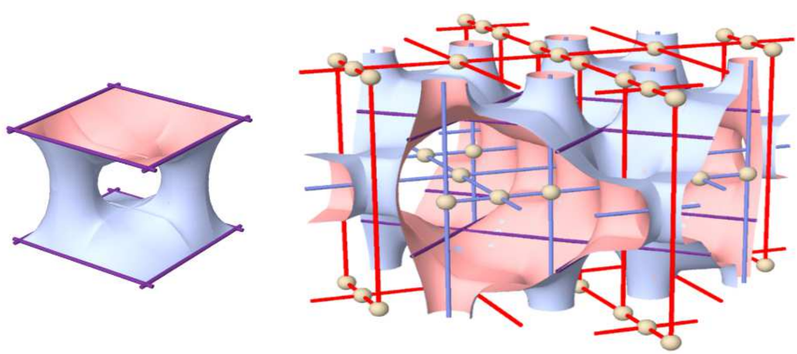


Figure 159 - A cell with tetragonal symmetry and two sleeves diagonal from each other and the surface formed by it

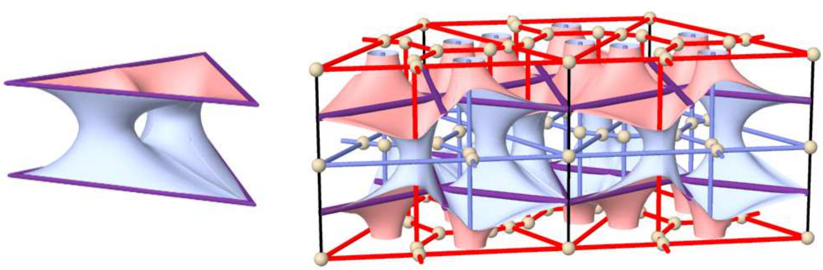


Figure 160 - A cell with hexagonal symmetry and two sleeves and the surface formed by it

5.7 Surface which partition space into two identical subspaces, for which the dual tunnel networks contain curved edges

Within the above mentioned "multi-sleeved class", surfaces of which dual tunnel network contain curved edges were also discovered. In this chapter, we show examples of two of these surfaces.

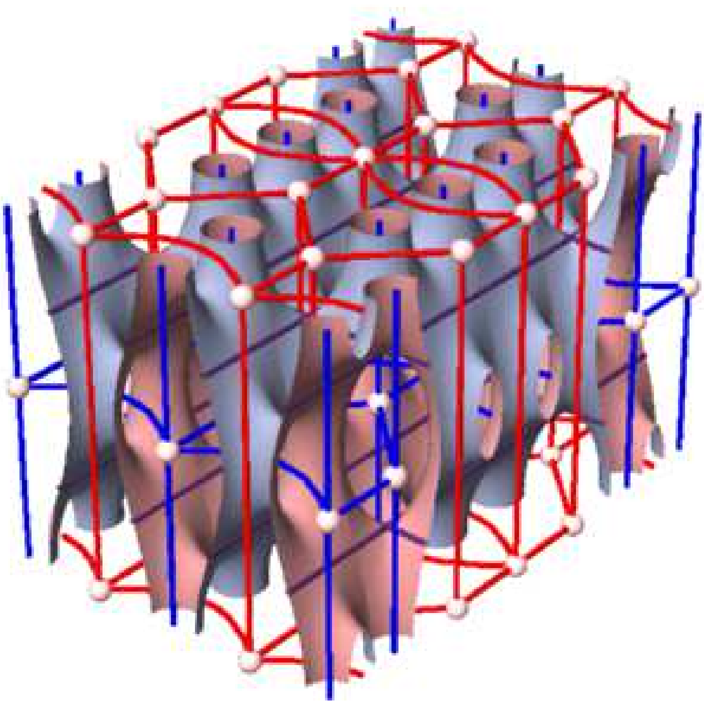


Figure 161 - A smooth surface which divides space into two identical subspaces

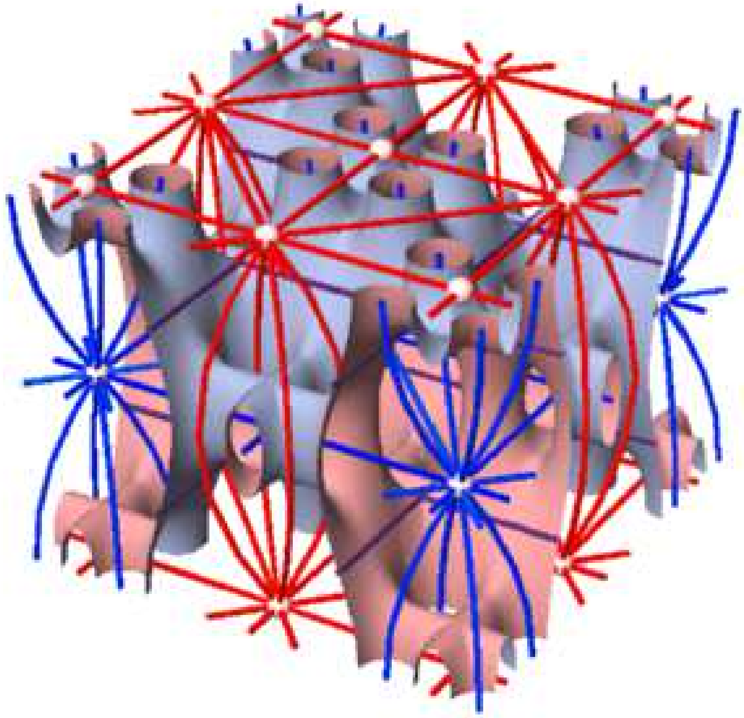


Figure 162 - Another example with both curved edges and straight edges

5.8 Identical dual networks, in which an inversion point transforms one into its dual-complementary

Through empirical search for unique networks, such as uniform networks (That is, a network which have uniform vertices and edges, such as the cubic network), one such unique network, in which each vertex is incident with three edges, was discovered (Figure 163)

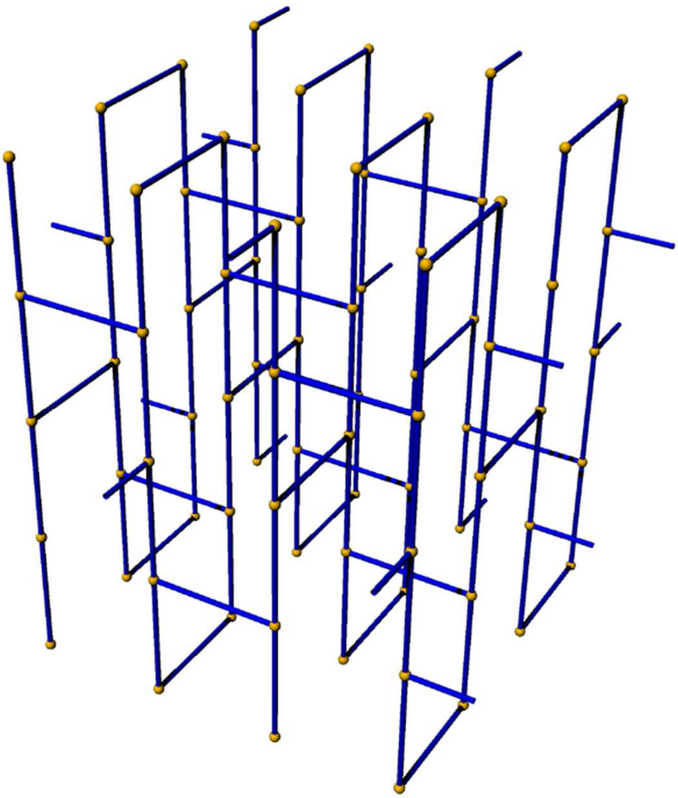


Figure 163 - A newly discovered uniform network

Locating the dual network, via locating the packing cell of the network (Figure 164), resulted in a dual network that is identical to the original network. Meaning, the two networks are dual-complementary.

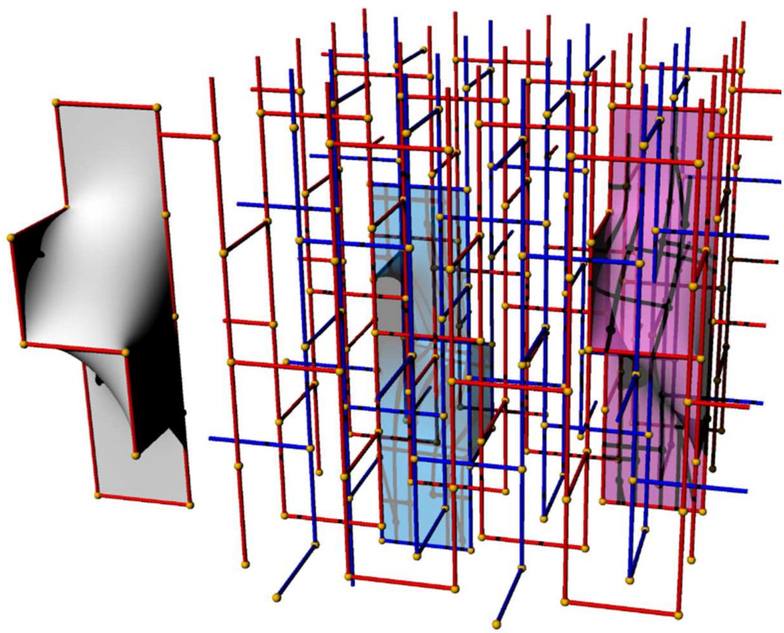


Figure 164 - Packing cell of the newly discovered uniform network and the dual-complementary networks

Unexpectedly, no 2-fold axis which rotate one network into its dual-complementary was found.

In this case, it was discovered that the symmetry operation which transforms one network into its dual-complementary is an inversion point symmetry.

Between the two networks is a smooth surface which partitions space into two identical subspaces (Figure 165), as exists between any two identical, dual-complementary networks. The elementary surface segment in this case is not bounded by a perimeter made of 2-fold axes, as in the two surface classes previously discussed.

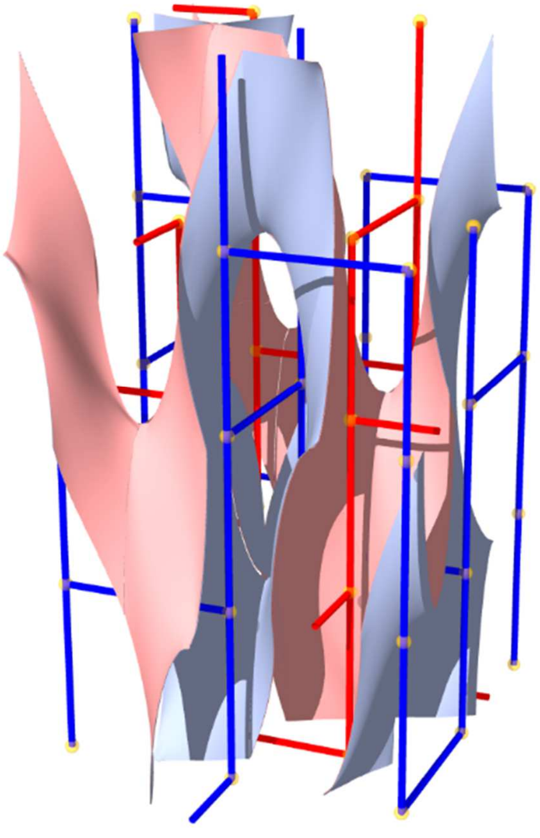


Figure 165 - The surface separating the dual-complementary networks

From this, we arrive at the conclusion that another class of surfaces which partition space into two identical subspaces exists. We refer to this class as the “inversion point surface class”. This discovery opens the door for farther investigation, and the development of a methodology for locating farther networks belonging to this new class.

6 Notation

6.1 Notation methods

> **God called the light "day," and the darkness he called "night."… God called the vault "sky."… God called the dry ground "land," and the gathered waters he called "seas." And God saw that it was good… (Genesis 1)**

We usually describe the different morphological phenomena along with a graphical representation. Some of the phenomena are easy to describe. However, as a phenomenon is more complex, so it is more difficult to describe it. A phenomenon which cannot be described is as good as one which does not exist. It is therefore desirable that there is an agreed upon method to describe these phenomena as accurately as possible, whether by naming them or by using accepted symbols.

Some of the morphological-geometric phenomena have existing well-known names. Almost anyone knows what a triangle, quadrilateral, pentagon, hexagon, etc. are. This is a notation based on the number of edges of the polygon. This information, however, is partial. The word "triangle" describes all polygons with three edges. And yet, it does not provide information regarding the length ratios between the edges. In order to provide more information, we must be more specific in the names used, such as equilateral, isosceles, right angle, etc.

Three-dimensional phenomena, such as a ball, cube, prism, pyramid, etc. are known by the names given to them. The Platonic solids, tetrahedron, octahedron, hexahedron (Cube), dodecahedron, and icosahedron, are regular objects with uniform edges vertices and faces. These objects are named after the number of faces they each have, without adding any information about the type of faces. And yet, because the number of these objects is small, it is easy to remember them by their names.

The Archimedean solids are of a higher complexity level than the Platonic solids. They have uniform vertices and edges, but not uniform faces. As their number is greater, their notation is likewise more complex. Some of the Archimedean solids have names, but most are known by an accepted notation. This notation is based on the fact that the faces which form these solids are regular, and additionally, their vertices are identical. That is, the arrangement of faces around a vertex is identical for all vertices. Describing a vertex, for instance Triangle-Square, describes a solid in which each vertex is incident with a triangle and two squares. This notation is usually shortened into numbers (3.4.4) in which each number describes the number of edges of a face, and the order of the numbers describes the order of the faces around the vertex. We generally begin with the face with the least number of edges. When two adjacent faces are identical, there is an even shorter notation, such as 3.4^2. The exponent represents the number of identical adjacent faces incident with the vertex.

The infinite edge uniform and vertex uniform polyhedra are described in the same way the finite polyhedra are described. This notation is topological. We know that the faces are regular, the edges are all equal, and that the number of faces and their order is the same between any two vertices. Yet there is no information about the absolute size of the edges.

Phenomena like non-regular solids, networks, surfaces, etc. do not have a notation. A small number of those have a specific name.

6.2 Notation method for surfaces which partition space into two identical subspaces

Each of the surfaces which partition space into two identical subspaces may be represented as an infinite polyhedra with saddle-shaped face. The surfaces are formed from periodic polygonal cells, and surface segments bounded within them. Since the surfaces are formed from identical periodic cells, it is natural to think that describing the cell is sufficient for describing the surface. However, while the surfaces are made from identical periodic units, the number of units which are incident with a vertex is not generally uniform. That is, there may exist, in a surface, different types of vertices. Another issue is that different periodic cells may form surfaces which are topologically similar.

The unique topological identification of a surface is based on the elementary translation cell of the surface, or based on the tunnel networks which are separated by the surface. Describing one of those two phenomena may be sufficient to describe the surface.

The elementary translation unit is a segment of the minimal surface. The only information we can provide about the elementary translation unit is the Euler genus. While this is a vital detail, it is not unique. There may be topologically different surfaces with an identical Euler genus.

A unique notation of the dual-complementary tunnel networks may be a tool for identifying topologically different surfaces.

Periodic networks may be represented as a tight packing of periodic solids of which edges form the edges of the network. A notation for these solids may be a sufficiently accurate tool for describing the network.

These solids may take several forms:

a. Platonic or Archimedean solids.

b. Solids with saddle-shaped faces and uniform vertices.
c. Solids with saddle-shaped faces and several types of vertices.
d. Hybrid solids, with both planar and saddle-shaped faces.

The planar faces may be regular or have different length edges.

At this point we will only describe visually how each packing cells of the network looks. A unique formal notation does not yet exist.

6.3 Description of the packing cells representing the surfaces

This chapter contains the description of the packing cells of the eight dual-complementary tunnel networks which are partitioned by soap solution surfaces.

The cubic surface (Figure 166) separates two cubic networks.

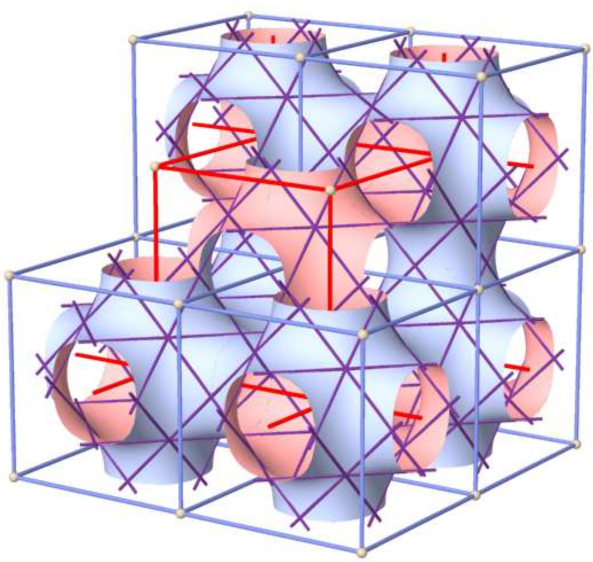


Figure 166 - The cubic surface

The packing cell of each of the dual-complementary networks is a cube (Hexahedra) (Figure 167).

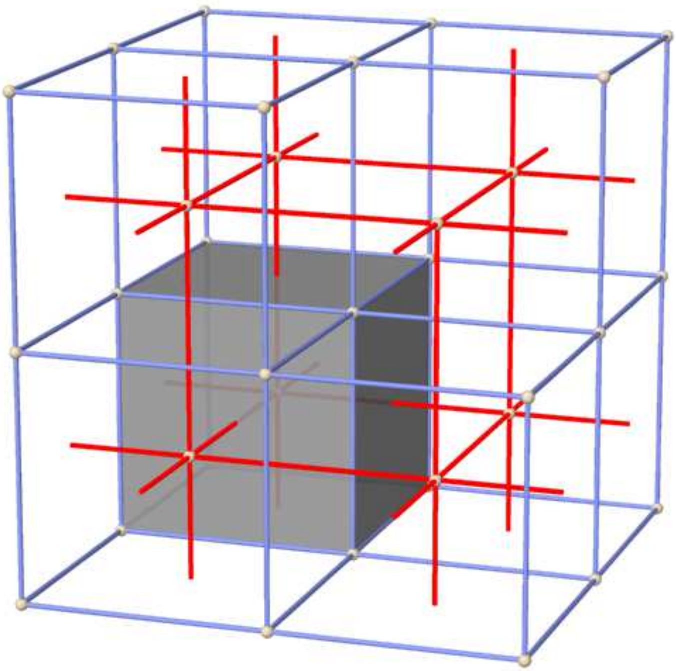


Figure 167 - The packing cell of the cubic network

The diamond surface (Figure 168) separates two diamond networks.

The packing cell consists of four saddle-shaped hexagonal faces (Figure 169). It has two types of vertices. One is incident with two faces. The other is incident with three faces.

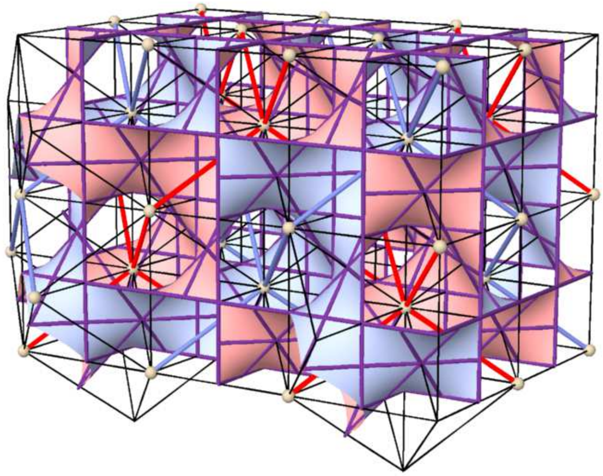


Figure 168 - The diamond network surface

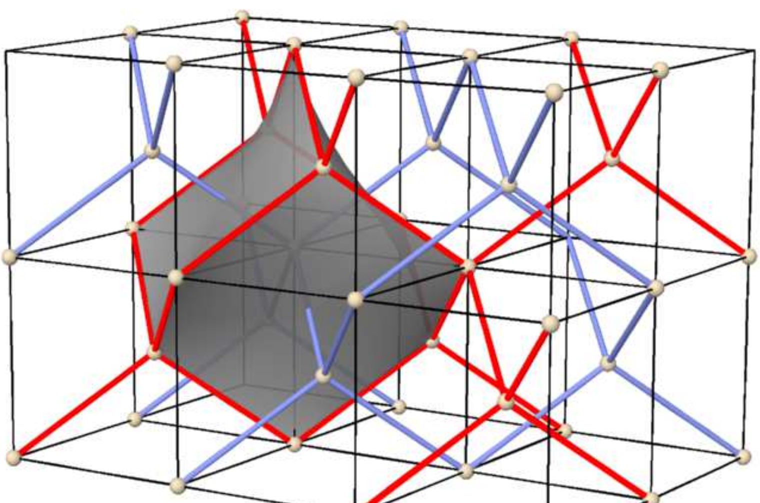


Figure 169 - The packing cell of the diamond network

The crosses network surface (Figure 170) separates two crosses networks.

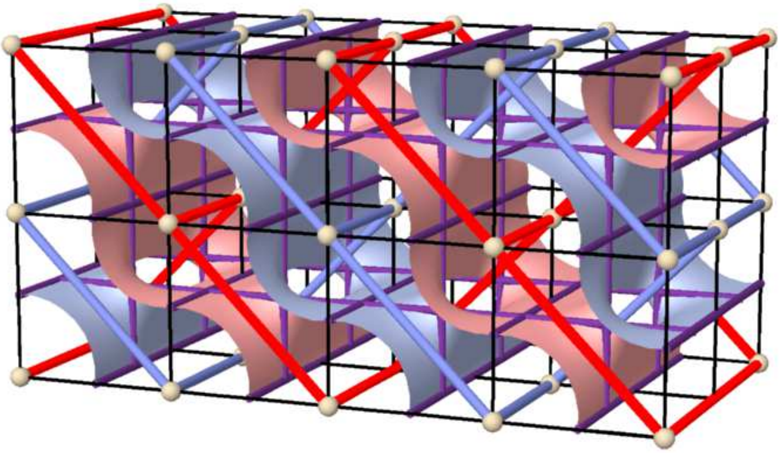


Figure 170 - The crosses network surface

The packing cell has two types of faces (Figure 171). Two faces are saddle-shaped octagons with right angles. The other two faces are planar surfaces with six edges which trace squares. It has two types of vertices. One coincides with two faces, one of each type. The other coincides with three faces, two of which are octagons.

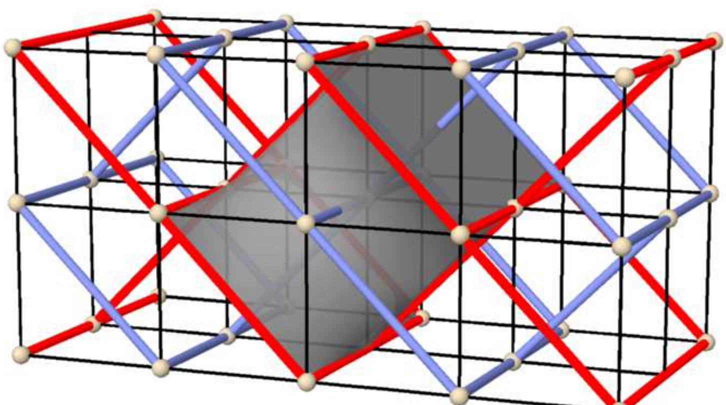


Figure 171 - The packing cell of the crosses network

The diamond cubic surface (Figure 172) separates two networks which are a combination a diamond network and a cubic network.

The diamond cubic network has two types of vertices, and therefore two types of packing cells (Figure 173). One consists of four saddle-shaped pentagons, with eight vertices, each being incident with three pentagons. The other has six faces, four of which are saddle-shaped pentagons, and two of which are planar rhombuses. It has ten vertices, two of which are each incident with two saddle-shaped pentagons, and the other eight are each incident with two saddle-shaped pentagons and one planar rhombus.

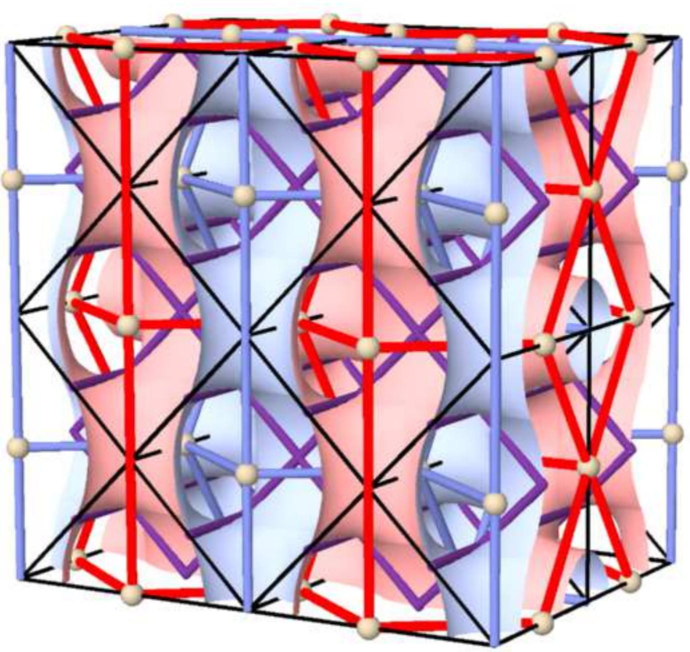


Figure 172 - The diamond cubic surface

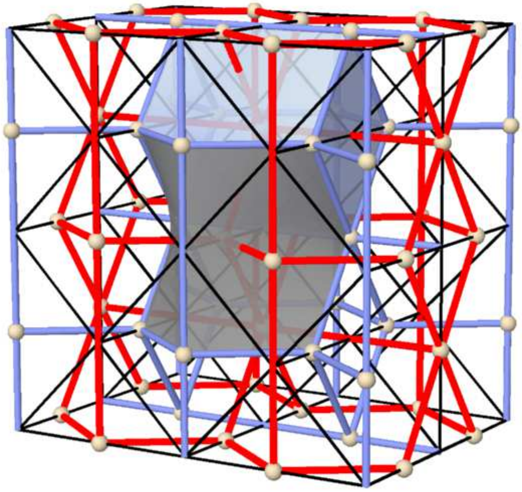


Figure 173 – The packings cell of the diamond cubic network

The 45^0, 90^0, 45^0 triangle surface (Figure 174) separates two networks which have two types of vertices.

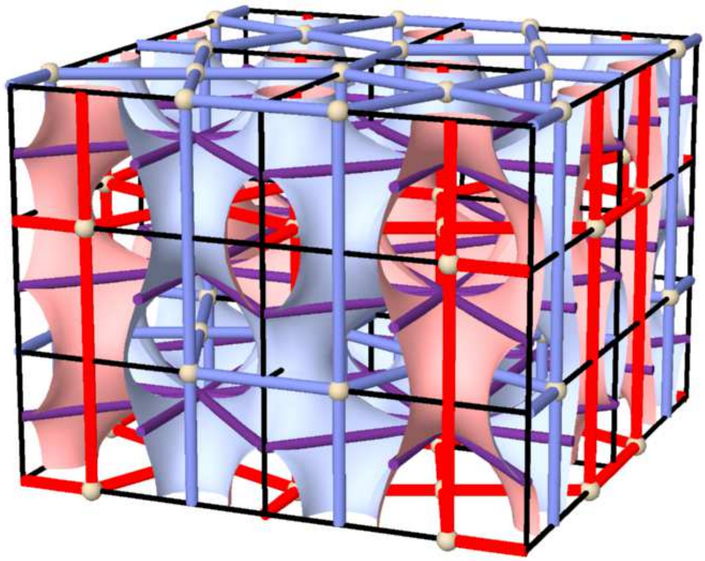


Figure 174 - The 45^0, 90^0, 45^0 triangle surface

They therefore have two packing cells (Figure 175). One consists of four saddle-shaped hexagons and ten vertices. Two of the vertices are incident with four faces, and the other eight are incident with two faces. The other cell consists of three types of faces. Two if its faces are saddle-shaped pentagons, two are planar non-regular pentagons, and one is a rectangle. It has ten vertices in total.

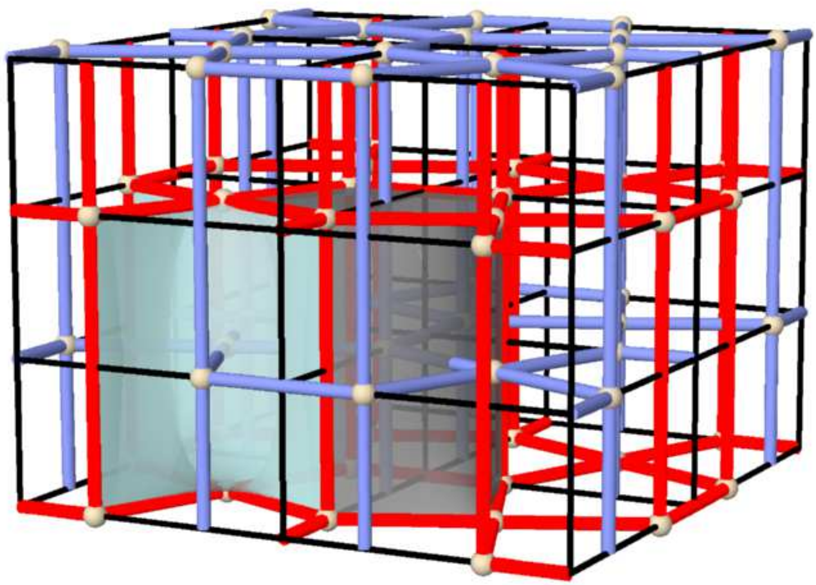


Figure 175 - The packing cells of the 45^0, 90^0, 45^0 triangle network

The face centered surface (Figure 176) separates two face centered networks, which have two types of vertices.

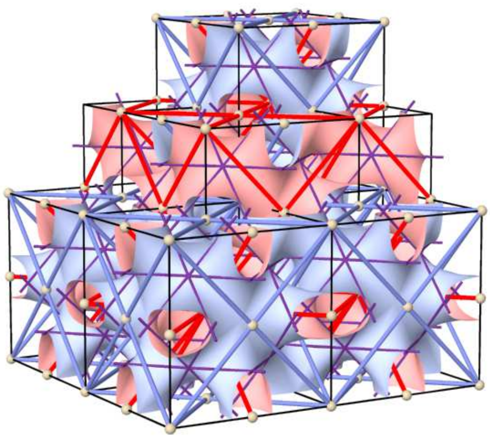


Figure 176 - The face centered surface

They therefore have two packing cells (Figure 177). One type of vertex is incident with twelve edges, and is at the center of a packing cell with twelve faces which are saddle-shaped rhombuses. We refer to this solid as a “hyperbolic rhombic dodecahedron”. The other type of vertex is incident with four edges, and is at the center of a packing cell with four faces which are saddle-shaped rhombuses. We refer to this solid as a “hyperbolic rhombic tetrahedron”.

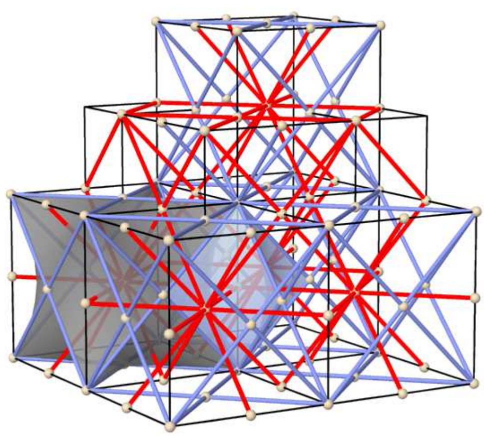


Figure 177 - The packing cells of the face centered network

The pentahedral-trihedral surface (Figure 178) separates two networks which have two types of vertices.

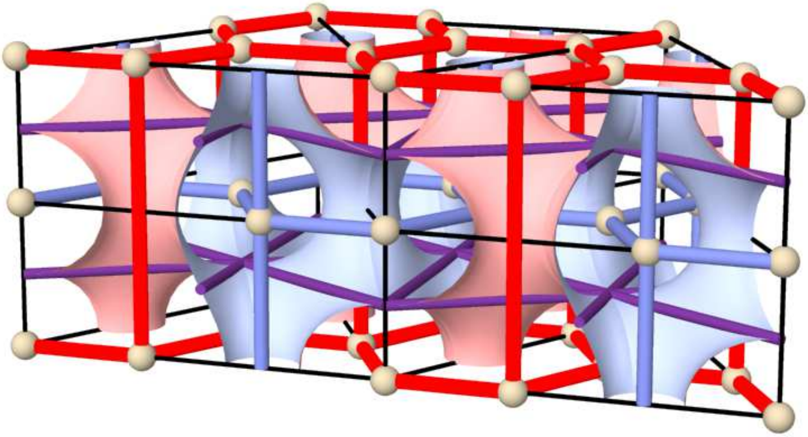


Figure 178 - The pentahedral-trihedral surface

They therefore have two packing cells (Figure 179). One type of vertex is at the intersection of three planar edges. It is at the center of a packing cell with three faces which are saddle-shaped hexagons. It has eight vertices, two of which are incident with three faces, and the rest are incident with two. The other type of vertex is the intersection of five edges, three of which are planar. It is at the center of a packing cell with five faces and twelve vertices. It has two types of faces, regular hexagons, and saddle-shaped hexagons. It also has two types of vertices, six are incident with a face of each type, and the other six are incident with one planar face and two saddle-shaped faces.

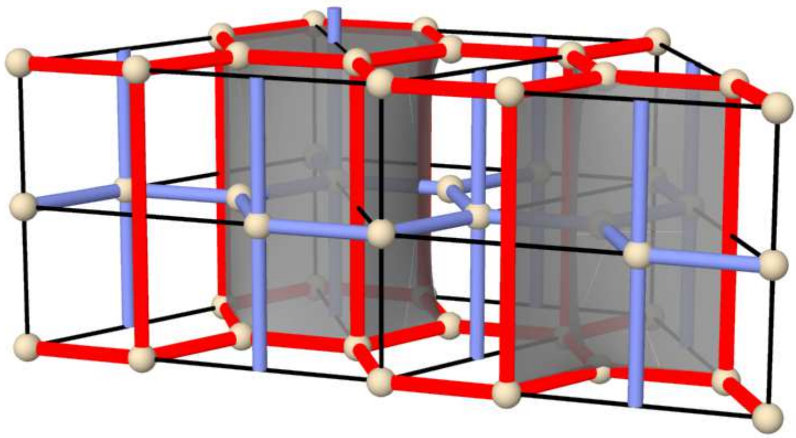


Figure 179 - The packing cells of the pentahedral-trihedral network

The 30^0, 60^0, 90^0 triangle surface (Figure 180) separates two networks which have three types of vertices.

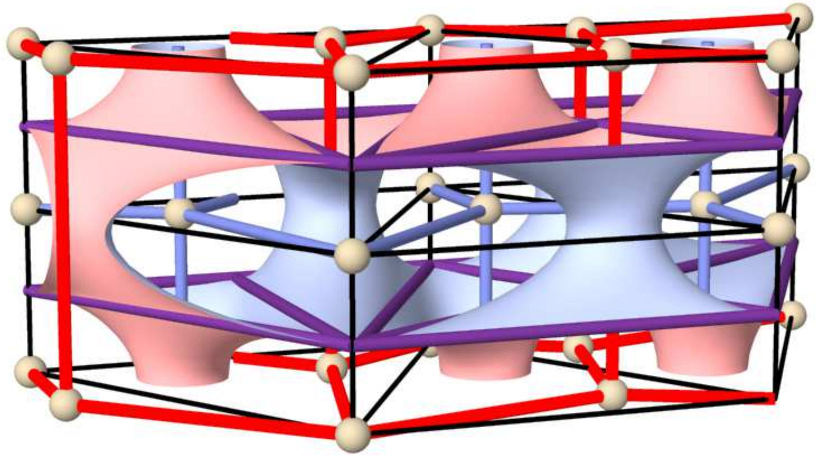


Figure 180 - The 30^0, 60^0, 90^0 triangle surface

They therefore have three packing cells (Figure 181). One type of vertex is at the intersection of six planar edges. It is at the center of a packing cell with six faces which are saddle-shaped hexagons.

Another type is at the intersection of three planar edges. It is at the center of a packing cell with three faces which are saddle-shaped hexagons which have a different shape from the faces of the other packing cell. The final type of vertex is incident with five edges, three of which are planar. It is at the center of a packing cell with five faces of four different types. Two of its faces are non-regular planar pentagons. One face is a saddle-shaped hexagon similar to the first packing cell. One is a saddle-shaped hexagon similar to the second packing cell. And the final face is a rectangle. The packing cells of this network have saddle-shaped hexagonal faces which are topologically similar but have different shapes.

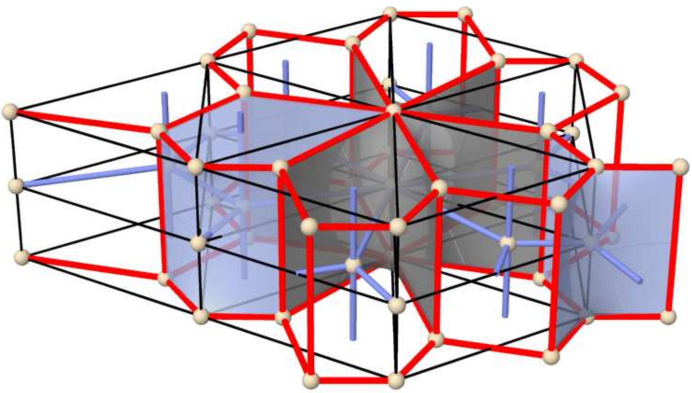


Figure 181 - The packing cells of the 30^0, 60^0, 90^0 triangle network

7 Summary

7.1 The method for searching and classifying the surfaces

The method for searching and classifying the surfaces proposed in this work is based on the properties of known surfaces which are periodic, infinite, and partition space into two identical subspaces.

The properties of the surfaces points towards a link between the surfaces and symmetry groups, and packing of space.

The surfaces being periodic means there is an elementary surface segment bounded by an elementary periodic region which fulfills all of the properties of the surface, and can be used to reproduce the entire surface.

The existence of Elementary Periodic Regions (E.P.R.) which represent all of the properties of a surface, as well as the existence of a replication process based on the symmetry groups in three-dimensional space, are the principles on which the method for searching and classifying the surfaces is built.

Such an E.P.R. which represents a surface which partitions space into two identical subspaces would contain the following elements:

a. Elementary surface segment – the smallest segment of the surface bounded by the E.P.R. which partitions it into two identical subspaces.
b. 2-fold axis or axes, which rotate the E.P.R. into itself. That is, they rotate one subspace into the other.
c. All of the vertex types of the dual-complementary tunnel networks, and the edges connecting them. The representation for each of the tunnel networks within the E.P.R. would be on one side of the elementary surface segment.

Filtering down the E.P.R.s which contain 2-fold axis or axes, out of all possible E.P.R.s, and replicating them leads to the discovery of networks, of which edges are all within infinite line representing 2-fold axes. These networks are called “2-fold axis networks”.

The method for locating the 2-fold axis networks should yield all such networks. The number of E.P.R.s which may contain 2-fold axis or axes is finite. This conclusion is based on the fact that the set of E.P.R.s which contain 2-fold axis or axes is a subset of the set of E.P.R.s which represent all symmetry groups, the number of which is also finite (32 in total).

Since geometrically identical E.P.R.s may represent different symmetry groups, the actual number of E.P.R.s is smaller.

The 2-fold axis networks, which form from replicating the E.P.R.s which contain 2-fold axes, are periodic, and may contain a periodic unit that is different from the network segment bounded by the E.P.R. In this work, we have detected within the 2-fold axis networks periodic cells which are formed by axis segments within the network, and in which a periodic surface segment is bounded. Since the 2-fold axis networks are fully contained within the surfaces which partition space into two identical subspaces, the surface segment bounded within the periodic cell of the 2-fold axis network is a periodic unit, replicating which would lead to the formation of the entire surface.

The surface segment which is bounded within the periodic cell of the 2-fold axis network is smooth and complete, in the sense that it is impossible to go from one side of the surface segment to the other without crossing the boundaries of the periodic cell.

The periodic cells of the 2-fold axis networks appear in two forms:

a. A surface segment bounded by a connected three-dimensional polygon
b. A surface segment bounded by two planar polygons. Such a cell is called a “split polygon cell”.

Within the periodic cells of the first form, topologically, only one type of surface segment may be bounded. In this work, the surface segment was represented as a minimal surface. That is, a “soap solution surface” which could be formed by dipping the connected three-dimensional polygon in a soap solution.

In the latter form, there are two ways for a surface segment to be bounded. One is a soap solution surface bounded by the two polygons. The other is smooth surface which contains sleeve-like segments connecting the two polygons. This form of surface bounded by split polygon cells leads to the discovery of a new class of surfaces which partitions space into two identical subspaces. This class of surfaces is referred to as the “multi-sleeved class of surfaces”.

The topological categorization of surfaces is done using the tunnel networks formed by the surfaces. That is, surfaces would be considered topologically distinct if and only if their tunnel networks are topologically distinct.

The periodic cells which contained soap solution surface segments yielded eight topologically distinct surfaces which partition space into two identical subspaces. These eight surfaces are unique, and are therefore included in a unique class called the “soap solution class of surfaces”.

The other form for a surface segment to be bounded by a split polygon cell leads to the discovery of the multi-sleeved class. This new class of surfaces contains an infinite number of surfaces, which are topologically distinct, and which partition space into two identical subspaces. **The implication of this is that there is an infinite number of distinct dual-complementary networks in three-dimensional space.**

7.2 Notation of the surfaces

In this work, a notation method for the surfaces which partition space into two identical subspaces was suggested, based on their tunnel networks. The choice to use the tunnel networks for notation is based on the fact that those are used for the topological classification of the surfaces.

Notation for the networks could be done in one of two methods:

a. Based on the vertices of the network: A notation for the different vertices, and the relation between them, may provide some information about the networks, but is insufficient for reconstructing them.
b. Based on the packing solids of the network: A notation which provides accurate information about the packing solids of the network, of which edges form the edges of the network, provides complete information which allows reconstructing the network, and describing it.

In examining the two methods, we have found that the second alternative provides the most information. The information provided by this notation is as follows:

a. The number of different types of packing solids. That is, the number of different types of vertices in the network.
b. The types of faces, and the number of faces incident with a vertex.
c. The number of vertices of each of the packing solids.
d. The number of edges of each of the solids, which can be derived from the notation.

e. The number of faces of each solid, which can be derived from the number of vertices and edges of the solid.

This information, which is provided by the suggested notation, should describe the geometric shape of the packing solids, and hence, the tunnel networks. With that available, locating the surface separating them is simple.

This notation method was demonstrated by providing the notation for the eight tunnel networks separated by the eight surfaces belonging to the soap solution class.

As the networks become more complex, meaning, the number of different types of vertices grows, so does the notation become more cumbersome. And as the types of faces increases, it becomes harder to reconstruct the packing solids.

This notation method suffers from several weak points:

a. The description of non-regular faces is inaccurate. The method describes the number of edges of each of the non-regular faces, but does not provide information about the relations between the edges, in length ratio or angles.
b. It is possible for geometrically distinct faces with the same number of edges to exist within the same solid. For instance, two three-dimensional hexagons which are different from one another.
c. In solids with more than one type of vertices, the notation does not provide information about the order in which they appear. When the number of different types of vertices is not large, it is still possible to reach the described network. As the number of types of vertices within a solid grows, the difficulty to reconstruct the packing solid using the suggested notation method increases.

Improving the method, or finding a better and clearer notation, is a worthy goal for farther research. The better the notation for surfaces and networks is, the easier it will be to describe them even to those not operating directly within the field.

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